Elliptic regularity theory applied to time harmonic Maxwell's equations

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(Joint work with Yves Capdeboscq)

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Introduction

(Time harmonic) Maxwell's equations have the form

$$\begin{cases} \nabla \times H = i\varepsilon E + J_e & \text{in } \Omega, \\ \nabla \times E = -i\mu H & \text{in } \Omega, \\ E \times \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

where

- $\Omega \subseteq \mathbb{R}^3$: $C^{1,1}$ bounded domain;
- $E, H \in H(\mathrm{curl}, \Omega) = \{u \in L^2(\Omega; \mathbb{C}^3) : \nabla \times u \in L^2(\Omega; \mathbb{C}^3)\}$: electric and magnetic fields;
- $\varepsilon, \mu \in L^\infty\left(\Omega; \mathbb{C}^{3\times 3}\right)$ with uniformly positive definite real parts: electric permittivity and magnetic permeability;
- $J_e \in L^2(\Omega; \mathbb{C}^3)$, $\mathrm{div} J_e = 0$: current source.

Problem

What assumptions on ε and μ imply

1. $E, H \in W^{1,2}(\Omega)$?
2. $E, H \in C^{0,\alpha}(\Omega)$?

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Maxwell's equations $\longrightarrow$ coupled elliptic system

$$\begin{cases} \nabla \times H = i\varepsilon E + J_e & \text{in } \Omega, \\ \nabla \times E = -i\mu H & \text{in } \Omega, \end{cases}$$

These equations can be easily rewritten as a coupled elliptic system (Leis, 1986):

$$\begin{cases} -\mathrm{div}\,(\varepsilon \nabla E_k) = \mathrm{div}\big((\partial_k \varepsilon)\, E + \varepsilon\,(\mathbf{e}_k \times i\mu H)\big) & \text{in } \Omega, \\ -\mathrm{div}\,(\mu \nabla H_k) = \mathrm{div}\big((\partial_k \mu) H - \mu\,(\mathbf{e}_k \times (J_e + i\varepsilon E))\big) & \text{in } \Omega. \end{cases}$$

As $E_k, H_k \notin W^{1,2}$, these equations have to be interpreted in a "very weak" sense:

$$\int_\Omega E_k \mathrm{div}\left(\varepsilon^T \nabla \bar{\varphi}\right) dx = \int_{\partial\Omega} (\partial_k \bar{\varphi}) \varepsilon E \cdot \nu \, ds + \int_\Omega ((\partial_k \varepsilon) E + \varepsilon\,(\mathbf{e}_k \times i\mu H)) \cdot \nabla \bar{\varphi}\, dx,$$

for any $\varphi \in W^{2,2}(\Omega; \mathbb{C})$.

Lemma ("very weak" implies "weak" - L^p theory for elliptic equations)

Suppose $\varepsilon \in C^0$. Take $p \in [6/5, \infty)$, $u \in L^2 \cap L^p$ and $F \in L^p$. If

$$-\mathrm{div}(\varepsilon \nabla u) = \mathrm{div} F \qquad \text{in } \Omega$$

in a very weak sense, then $u \in W^{1,p}$.

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- By the Lemma we obtain $E_k, H_k \in W^{1,2}$, namely $E, H \in W^{1,2}$.
- By Sobolev embedding $E, H \in L^6$, whence $F \in L^6$.
- Finally, by De Giorgi-Nash we obtain $E, H \in C^{0,\alpha}$.

It seems that the $W^{1,\infty}$ assumption is necessary to obtain F$\in L^2$.

Is it possible to improve this result by using the L^p theory?

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$$\begin{cases} -\mathrm{div}\left(\varepsilon\nabla E_k + \xi\nabla H_k\right) = \mathrm{div}\left((\partial_k\varepsilon)E + (\partial_k\xi)H + i\varepsilon\left(\mathbf{e}_k \times (\zeta E + \mu H)\right)\right) \\ \qquad\qquad + \mathrm{div}\left(-\xi\left(\mathbf{e}_k \times (i\varepsilon E + i\xi H + J_e)\right)\right) \text{ in } \Omega. \\ -\mathrm{div}\left(\zeta\nabla E_k + \mu\nabla H_k\right) = \mathrm{div}\left((\partial_k\zeta)E + (\partial_k\mu)H - \mu\left(\mathbf{e}_k \times (i\varepsilon E + i\xi H + J_e)\right)\right) \\ \qquad\qquad + \mathrm{div}\left(\zeta\left(\mathbf{e}_k \times (i\zeta E + i\mu H)\right)\right) \text{ in } \Omega. \end{cases}$$

  We get: if $\varepsilon, \xi, \zeta, \mu \in W^{1,3+\delta}$ then $E, H \in C^{0,\alpha}$.
- Higher regularity results can be easily obtained.

Conclusions

Past

- The regularity theory for Maxwell's equations has been studied mainly with ad-hoc techniques
- The assumption $\varepsilon \in W^{1,\infty}$ was believed to be optimal to have $E \in C^{0,\alpha}$

Present

- The L^p theory for elliptic equations can be easily applied to Maxwell's equations
- Main result: $\varepsilon, \mu \in W^{1,3+\delta} \implies E, H \in C^{0,\alpha}$
- Same result for bianisotropic materials, no need to develop a different approach

Future

- Is $W^{1,3+\delta}$ the optimal assumption? Probably not! (ongoing work with Jan Kristensen)

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Thank you for your attention!

G. S. Alberti and Y. Capdeboscq.
Elliptic regularity theory applied to time harmonic anisotropic Maxwell's equations with less than Lipschitz complex coefficients.
Siam J. Math. Anal., to appear.