Using multiple frequencies to enforce non-zero constraints in PDE and applications to hybrid imaging problems

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Outline of the talk

1. Introduction to hybrid imaging and non-zero constraints
2. Using multiple frequencies to enforce non-zero constraints
3. Additional results

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Motivation: quantitative hybrid imaging problems

- Microwave imaging + ultrasounds [Triki, 2010, Ammari et al., 2011]

$$\begin{cases} \operatorname{div}(a\,\nabla u^i_\omega) + \omega^2\,\varepsilon\,u^i_\omega = 0 & \text{in } \Omega, \\ u^i_\omega = \varphi_i & \text{on } \partial\Omega. \end{cases}$$

Problem: $a(x)\,\left|\nabla u^i_\omega\right|^2(x), \quad \varepsilon(x)\,\left|u^i_\omega\right|^2(x) \quad \xrightarrow{?} \quad a, \varepsilon$

- Quantitative thermo-acoustic [Bal et al., 2011, Ammari et al., 2013]

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Problem: $\sigma(x)\,\left|u^i_\omega\right|^2(x) \quad \xrightarrow{?} \quad \sigma$

- MREIT [Seo et al., 2012, Bal and Guo, 2013]

$$\begin{cases} \operatorname{curl} E^i_\omega = \mathbf{i}\omega H^i_\omega & \text{in } \Omega, \\ \operatorname{curl} H^i_\omega = -\mathbf{i}(\omega\varepsilon + \mathbf{i}\sigma)E^i_\omega & \text{in } \Omega, \\ E^i_\omega \times \nu = \varphi_i \times \nu & \text{on } \partial\Omega. \end{cases}$$

Problem: $H^i_\omega(x) \quad \xrightarrow{?} \quad \varepsilon, \sigma$

The measurements are meaningful at $x \in \Omega$ if at least $u^i_\omega(x) \neq 0$, $\nabla u^i_\omega(x) \neq 0$, ...

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Take $\varphi_1, \ldots, \varphi_{d+1}$, where d is the dimension.

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$$e_\omega^{i,j} = \sigma\, u_\omega^i \overline{u_\omega^j} \quad \xrightarrow{?} \quad \sigma$$

- $A_\omega = \left[\nabla \frac{e_\omega^{2,1}}{e_\omega^{1,1}} \quad \cdots \quad \nabla \frac{e_\omega^{d+1,1}}{e_\omega^{1,1}}\right]$ wherever $u_\omega^1 \neq 0$
- $|\det A_\omega| \geq c \left|\det \begin{bmatrix} u_\omega^1 & \cdots & u_\omega^{d+1} \\ \nabla u_\omega^1 & \cdots & \nabla u_\omega^{d+1} \end{bmatrix}\right|$
- $v_\omega = A_\omega^{-1} \operatorname{div}(A_\omega)^T$
- Exact formula for σ [Ammari et al., 2013, Bal and Uhlmann, 2013]

$$\sigma = \frac{-\Re v_\omega \cdot \Im v_\omega + \operatorname{div} \Im v_\omega}{2\omega}.$$

The constraints $|u_\omega^1| \geq C > 0$ and $\left|\det \begin{bmatrix} u_\omega^1 & \cdots & u_\omega^{d+1} \\ \nabla u_\omega^1 & \cdots & \nabla u_\omega^{d+1} \end{bmatrix}\right| \geq C$ give uniqueness, stability and explicit reconstruction of the unknown σ.

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The Helmholtz equation

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- $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$: smooth bounded domain
- $\varepsilon \in L^\infty(\Omega)$ such that $\Lambda^{-1} \leq \varepsilon \leq \Lambda$ in Ω
- $\sigma \in L^\infty(\Omega)$ such that either $\Lambda^{-1} \leq \sigma \leq \Lambda$ or $\sigma = 0$ in Ω
- $\omega \in \mathcal{A} = [K_{min}, K_{max}]$: admissible frequencies

- $K \subset \mathcal{A}$: finite set of frequencies
- $\varphi_1, \ldots, \varphi_{d+1}$: boundary conditions
- $K \times \{\varphi_i\}$: set of measurements

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Complete Sets of Measurements

A set of measurements $K \times \{\varphi_i : i = 1, \dots, d+1\}$ is *C-complete* if for every $x \in \Omega$ there exists $\bar{\omega}(x) \in K$ such that:

1. $\left|u^1_{\bar{\omega}}\right|(x) \geq C > 0$,
2. $\left|\det \begin{bmatrix} \nabla u^2_{\bar{\omega}} & \cdots & \nabla u^{d+1}_{\bar{\omega}} \end{bmatrix}\right|(x) \geq C > 0$,
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These constraints arise in various contexts:

- Microwaves + ultrasounds:
  - stability: need 1. [Triki, 2010]
  - reconstruction formulae: need 1., 2. and 3. [Ammari et al., 2011]
- Quantitative thermo-acoustics:
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- General elliptic equations (quantitative photo-acoustics, elastography):
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How can we construct complete sets of measurements, namely find K and φ_i?

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Several approaches

1. $\left|u^1_{\bar\omega}\right|(x) \geq C,$ 2. $\left|\det\left[\nabla u^2_{\bar\omega} \quad \cdots \quad \nabla u^{d+1}_{\bar\omega}\right]\right|(x) \geq C,$ 3. $\dots$

- Complex geometric optics solutions [Sylvester and Uhlmann, 1987]
  - $u^{(t)}_{\omega_0}(x) = e^{tx_m}\left(\cos(tx_l) + \mathbf{i}\sin(tx_l)\right)(1+\psi_t), \quad t \gg 1.$
  - If $t \gg 1$ then $u^{(t)}_{\omega_0}(x) \approx e^{tx_m}\left(\cos(tx_l) + \mathbf{i}\sin(tx_l)\right)$ in C^1 [Bal and Uhlmann, 2010]
  - The traces on the boundary of these solutions give the required 1., 2. and 3.
  - Need smooth coefficients, construction depends on coefficients.
- Runge approximation [Bal and Uhlmann, 2013]
  - There exist solutions that are locally closed to the solutions of the constant coefficient PDE.
  - Based on unique continuation, non constructive.
- Stability results without the constraints
  - Ultrasounds + microwave [Alessandrini, 2014], Quantitative photoacoustic tomography [Alessandrini et al., 2015]
  - Based on quantitative estimates of unique continuation.

Focus of this talk: new approach to construct suitable boundary conditions.

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  - $u_{\omega_0}^{(t)}(x) = e^{tx_m}\left(\cos(tx_l) + \mathbf{i}\sin(tx_l)\right)(1+\psi_t), \quad t \gg 1.$
  - If $t \gg 1$ then $u_{\omega_0}^{(t)}(x) \approx e^{tx_m}\left(\cos(tx_l) + \mathbf{i}\sin(tx_l)\right)$ in C^1 [Bal and Uhlmann, 2010]
  - The traces on the boundary of these solutions give the required 1., 2. and 3.
  - Need smooth coefficients, construction depends on coefficients.
- Runge approximation [Bal and Uhlmann, 2013]
  - There exist solutions that are locally closed to the solutions of the constant coefficient PDE.
  - Based on unique continuation, non constructive.
- Stability results without the constraints
  - Ultrasounds + microwave [Alessandrini, 2014], Quantitative photoacoustic tomography [Alessandrini et al., 2015]
  - Based on quantitative estimates of unique continuation.

Focus of this talk: new approach to construct suitable boundary conditions.

Outline of the talk

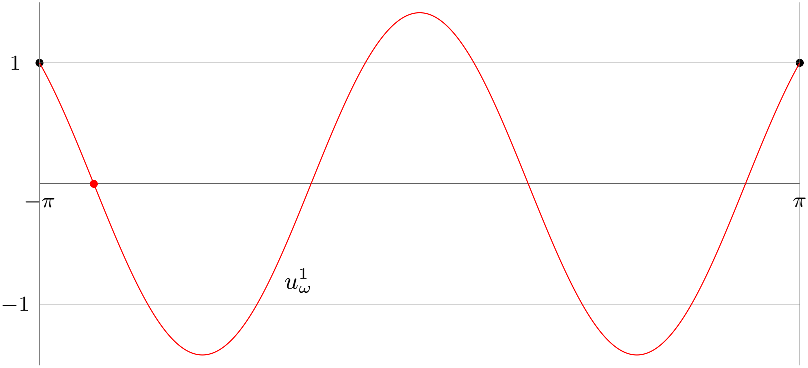
1. Introduction to hybrid imaging and non-zero constraints
2. Using multiple frequencies to enforce non-zero constraints
3. Additional results

Multi-Frequency Approach: basic idea I

As an example, let us consider the 1D case with $\varepsilon = 1$ and $\sigma = 0$.

1. $\left|u_\omega^1(x)\right| \geq C$: the zero set of u_ω^1 moves when ω varies:

$$\varphi(-\pi) = 1,\ \varphi(\pi) = 1$$



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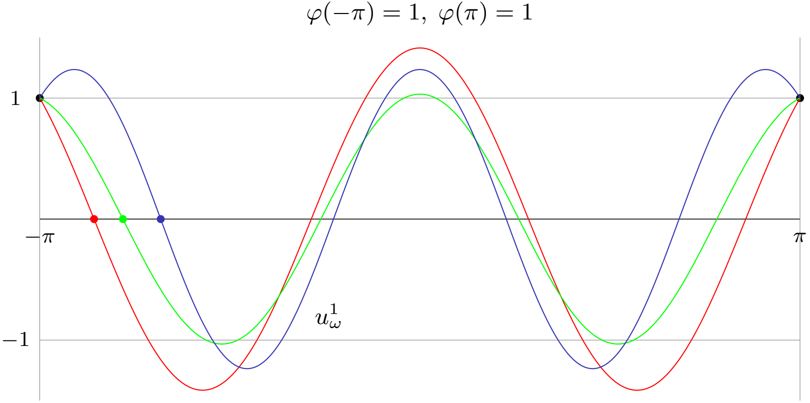

Multi-Frequency Approach: basic idea II

1. $\left|u_\omega^1(x)\right| \geq C$: the zero set of u_ω^1 may not move if the boundary condition is not suitably chosen:

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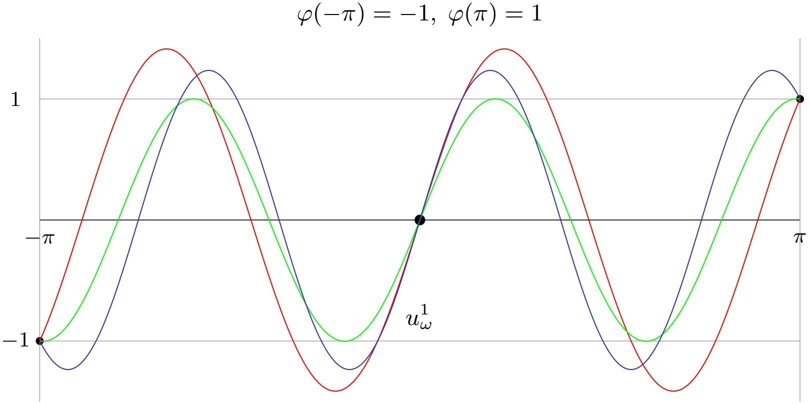

Multi-Frequency Approach: $\omega = 0$

1. $\left|u_0^1(x)\right| \geq C_0 > 0$ everywhere for $\omega = 0$ $\implies$ the zeros "move"

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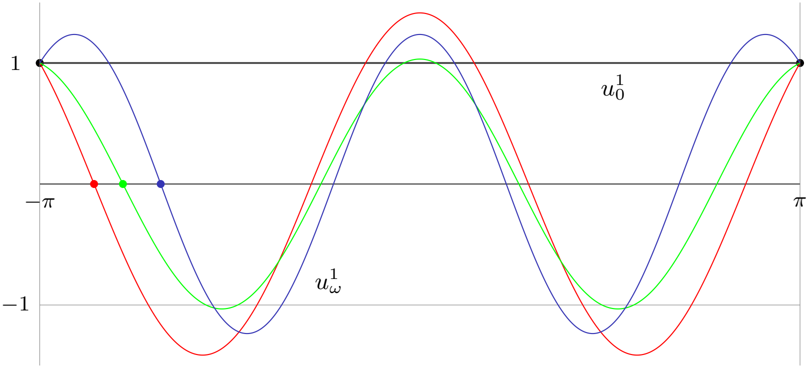

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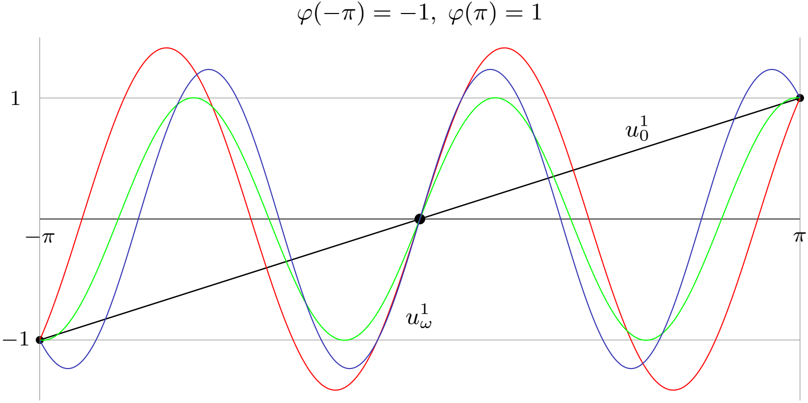

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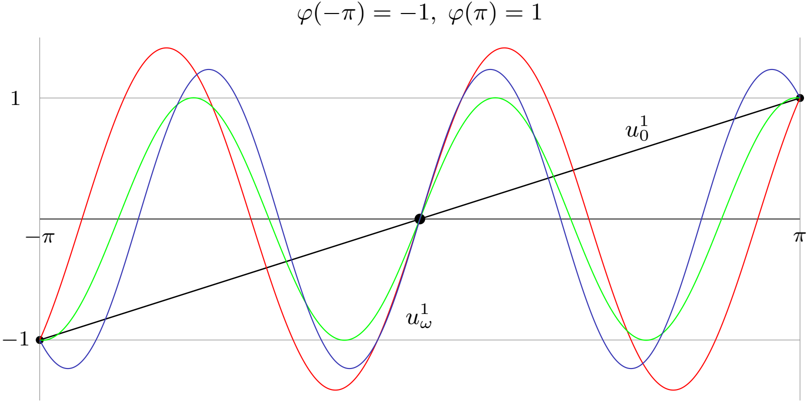

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What happens in $\omega = 0$?

$$\begin{cases} \Delta u_\omega^i + (\omega^2\varepsilon + \mathbf{i}\omega\sigma)\, u_\omega^i = 0 & \text{in } \Omega, \\ u_\omega^i = \varphi_i & \text{on } \partial\Omega. \end{cases}$$

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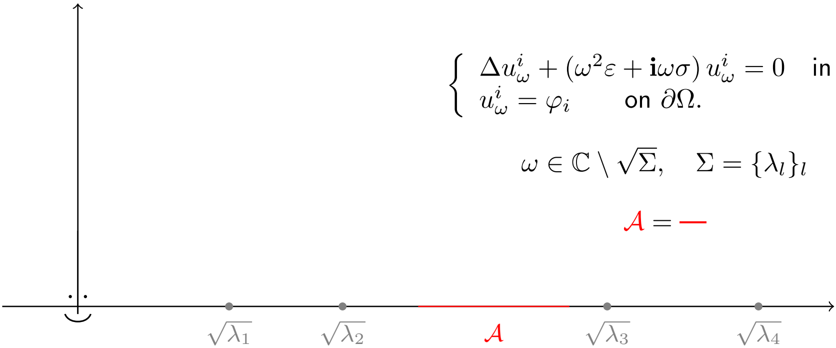

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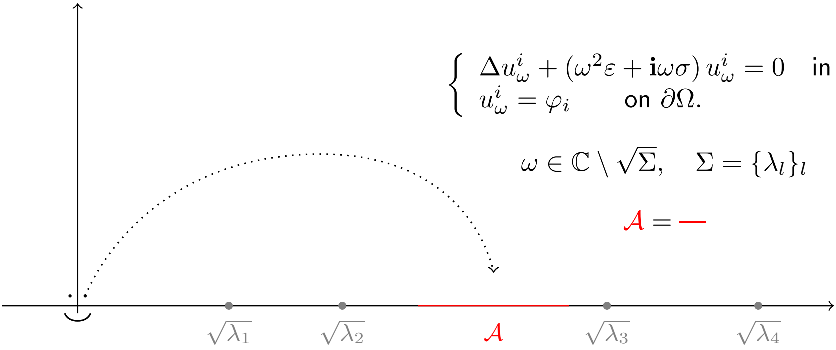

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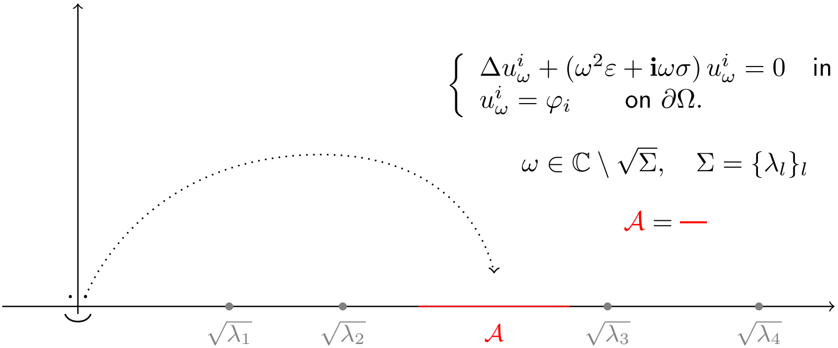

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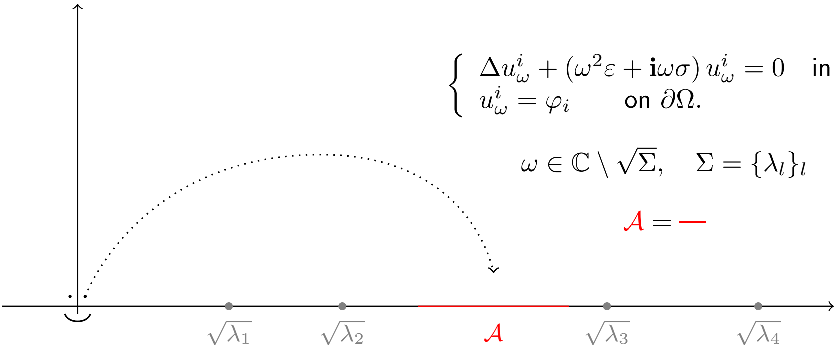

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There exist $C > 0$ and $n \in \mathbb{N}^*$ depending only on Ω, Λ and $\mathcal{A}$ such that

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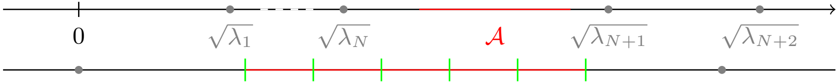

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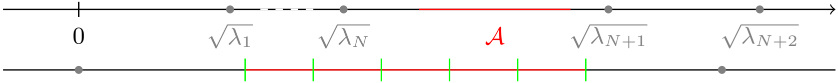

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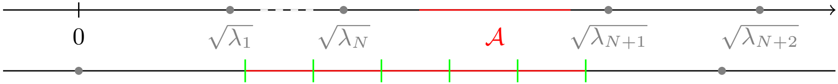

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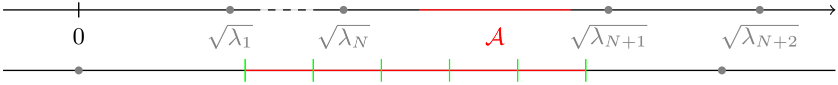

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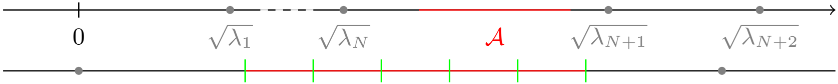

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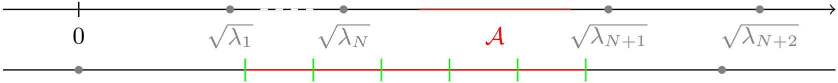

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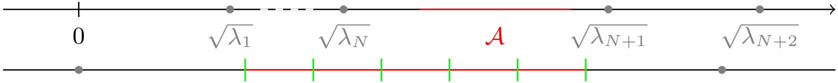

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There exist $C > 0$ and $n \in \mathbb{N}^*$ depending only on Ω, Λ and $\mathcal{A}$ such that

$$K^{(n)} \times \{1, x_1, \dots, x_{d+1}\}$$

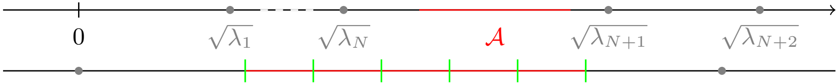
is C-complete.

- Consider for simplicity condition 1.
- Lemma [Momm, 1990]: Let g be a holomorphic function in $B(0, K_{max})$ such that $g(0) = 1$. There exists $\omega \in \mathcal{A}$ s.t. $|g(\omega)| \geq C(\mathcal{A}, \sup|g|) > 0$.
- Apply Lemma with $g_x(\omega) = u_\omega(x) \prod_{l=1}^{N} \frac{(\lambda_l - \omega^2)}{\lambda_l} \implies |g_x(\omega_x)| \geq C$.
- Finally, $\|\partial_\omega u_\omega\|_{C(\overline{\Omega})} \leq D$ gives a set $K^{(n)}$ for n big enough.

Main result

$K \times \{\varphi_i\}_i$ is *C-complete* if for all $x \in \Omega$ there exists $\bar{\omega} \in K$ s.t. 1 $\left|u^1_{\bar{\omega}}\right|(x) \geq C$,...

$K^{(n)}$: uniform partition of $\mathcal{A} = [K_{min}, K_{max}]$ with n points



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Outline of the talk

Another estimate on $\#K$

$K \times \{\varphi\}$ is *C-complete* if for all $x \in \Omega$ there exists $\bar{\omega} \in K$ s.t. $|u_{\bar{\omega}}^{\varphi}|(x) \geq C > 0$.

Theorem (Alberti and Capdeboscq, 2015)

Take $\varphi = 1$. Assume that σ and ε are real analytic. The set

$$\left\{(\omega_1, \ldots, \omega_{d+1}) \in \mathcal{A}^{d+1} : \min_{\overline{\Omega}}(\left|u_{\omega_1}^{\varphi}\right| + \cdots + \left|u_{\omega_{d+1}}^{\varphi}\right|) > 0\right\}$$

is open and dense in $\mathcal{A}^{d+1}$.
In other words, (almost any) $d+1$ frequencies give a complete set.

Proof.

- Classical elliptic regularity theory implies that u_{ω}^{φ} is real analytic
- The set $X = \{x \in \Omega : \left|u_{\omega_1}^{\varphi}\right| = \cdots = \left|u_{\omega_l}^{\varphi}\right| = 0\}$ is an analytic variety
- Stratification for analytic varieties: $X = \bigcup_p A_p$, A_p analytic submanifolds
- Use that $\{\omega : u_{\omega}^{\varphi}(x) = 0\}$ consists of isolated points (holomorphicity in ω)

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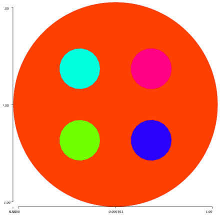
Numerical simulations on $\#K$

- Can we remove the analyticity assumption on the coefficients?
- A numerical test has been performed in 2D on 6561 different combinations of coefficients of the type



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$\#K = 2$	$\#K = 3$	$\#K \geq 4$
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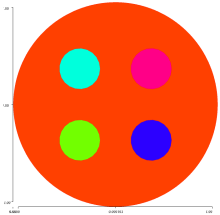

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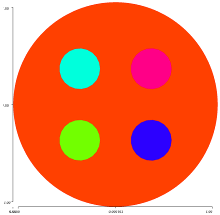

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Some generalisations

- There is no need to consider these particular non-zero constraints:

Let $b, r \in \mathbb{N}^*$ be two positive integers, $C > 0$ and let

$$\zeta = (\zeta_1, \dots, \zeta_r)\colon C^\nu(\overline{\Omega})^b \longrightarrow C(\overline{\Omega})^r \quad \text{be analytic.}$$

$K \times \{\varphi_1, \dots, \varphi_b\}$ is *(ζ, C)-complete* if for every $x \in \overline{\Omega}$ there exists $\bar{\omega} \in K$ s.t.

$$\left|\zeta_j\left(u_{\bar{\omega}}^1, \dots, u_{\bar{\omega}}^b\right)(x)\right| \geq C, \quad j = 1, \dots, r.$$

- In 2D, everything works with $a \in C^{0,\alpha}(\Omega; \mathbb{R}^{2\times 2})$ and

$$\operatorname{div}(a\,\nabla u_\omega^i) + (\omega^2\varepsilon + \mathbf{i}\omega\sigma)u_\omega^i = 0$$

by using the absence of critical points for the conductivity equation.

- Ammari et al. (2014) have successfully adapted this method to

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The assumption $a \approx 1$ in 3D seems necessary since the determinant of the gradients of solutions of the conductivity equation always vanishes [Briane et al., 2004]. However, the case $\omega = 0$ may not be needed for the theory to work:

$$\varphi(-\pi) = \alpha$$
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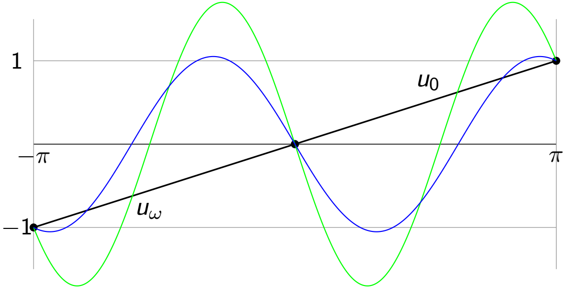

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Suppose $a, \varepsilon \in C^2(\mathbb{R}^3)$. For a generic C^2 bounded domain Ω and a generic $\varphi \in C^1(\overline{\Omega})$ there exists a finite $K \subseteq \mathcal{A}$ such that

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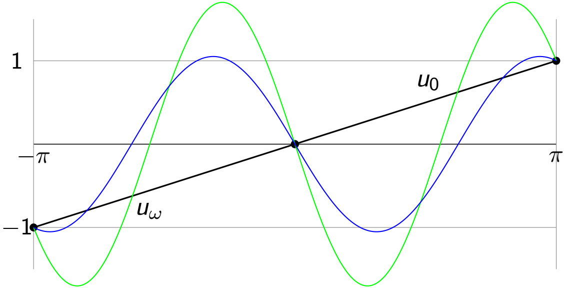

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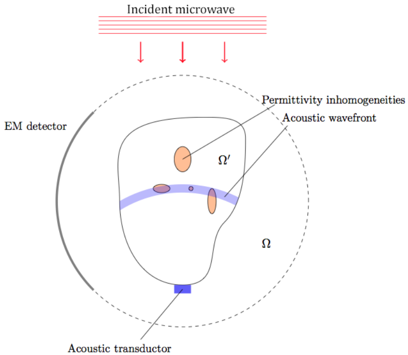

- Model

$$\begin{cases} \Delta u_\omega + \omega^2 \varepsilon u_\omega = 0 \text{ in } \Omega, \\ \frac{\partial u_\omega}{\partial \nu} - i\omega u_\omega = \varphi \text{ on } \partial\Omega. \end{cases}$$

- Internal data:

$$\psi_\omega = |u_\omega|^2 \nabla \varepsilon$$

- Linearised problem:

$$D\psi_\omega[\varepsilon](\rho) \mapsto \rho$$

In order to have well-posedness of the linearised inverse problem we need

$$\sum_{\omega \in K} \|D\psi_\omega[\varepsilon](\rho)\| \geq C\|\rho\|, \qquad \rho \in H^1(\Omega),$$

or equivalently $\cap_{\omega \in K} \ker D\psi_\omega[\varepsilon] = \{0\}$.

Theorem (Alberti, Ammari, Ruan, 2014) This holds true with multiple frequencies.

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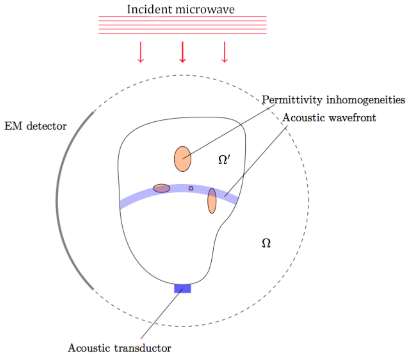

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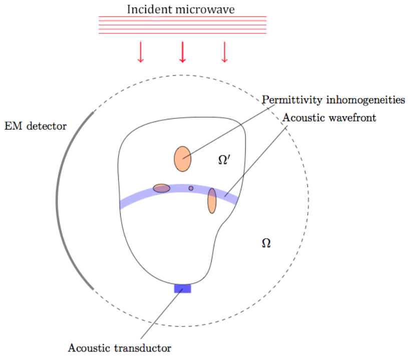

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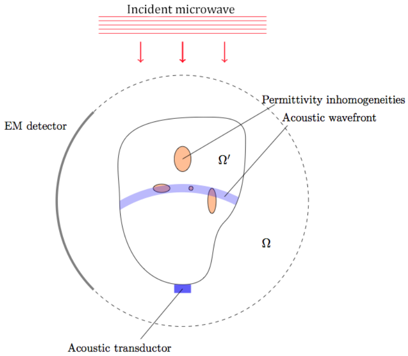

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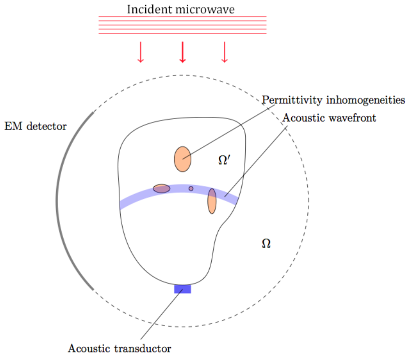

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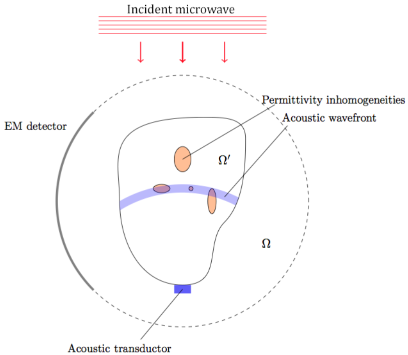

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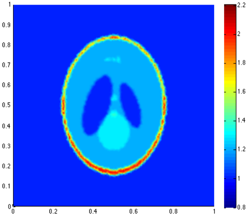

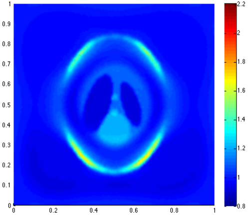

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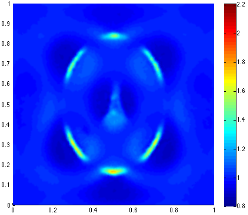
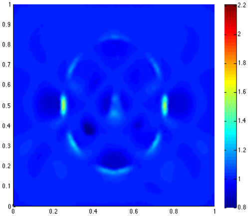


(b) $K = \{15\}$



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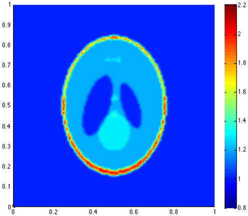

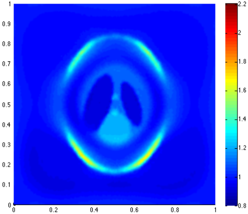

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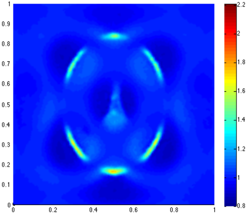
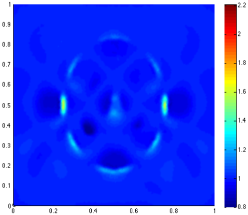
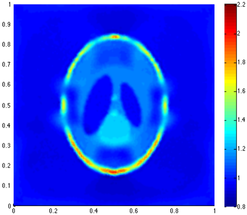


(b) $K = \{15\}$



(c) $K = \{20\}$



(d) $K = \{10, 15, 20\}$

Conclusions

Past

- In order to use the reconstruction algorithms for several hybrid techniques, we need to find illuminations such that the solutions of the Helmholtz equation (or Maxwell's equations) satisfy some non-zero constraints.
- These are classically constructed with complex geometric optics solutions or the Runge approximation.

Present

- We propose an alternative by using a multi-frequency approach:
  - A priori conditions on the illuminations which do not depend on the coefficients;
  - The coefficients do not have to be smooth;
  - A priori lower bounds and number of frequencies.
- Same method for Maxwell's equations.

Future

- We need $n = d + 1$ frequencies with real analytic coefficients. Can we drop this (very strong) assumption? (with Yves Capdeboscq)
- In 3D, can we drop the assumption $a \approx const$?

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