Two-dimensional reproducing formulae arising from the metaplectic representation and their discretisation

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Outline of the talk

1. Introduction to reproducing formulae
2. Two-dimensional metaplectic reproducing formulae
3. The discretisation of the reproducing formulae

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Representations of functions

Much harmonic analysis has focused on different ways of representing functions $f \in L^2(\mathbb{R}^d)$ by means of a superposition of building blocks, or atoms.

1. Gabor analysis: for $f \in L^2(\mathbb{R})$

$$f = \int_{\mathbb{R}^2} \langle f, T_b M_\xi \eta \rangle T_b M_\xi \eta \, db d\xi,$$

   and obvious extension to higher dimensions.

2. Wavelets: for $f \in L^2(\mathbb{R})$

$$f = \int_{\mathbb{R} \rtimes \mathbb{R}_+} \mathcal{W} f(b,a) \frac{1}{a^{1/2}} T_b D_{a^{-1}} \eta \, db \frac{da}{a^2},$$

   and extensions to higher dimensions with one dilation, tensor product of 1D wavelets on $L^2(\mathbb{R}^2) = L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ or directional wavelets with rotations.

3. Shearlets: for $f \in L^2(\mathbb{R}^2)$

$$f(x) = \int_{\mathbb{R}^2 \rtimes (\mathbb{R}_+ \times \mathbb{R})} \mathcal{S} f(b,a,s) a^{-3/4} \eta\Big(\left[\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right]^{-1} \left[\begin{smallmatrix} a & 0 \\ 0 & a^{1/2} \end{smallmatrix}\right]^{-1} (x-b) \Big) \, db \frac{da}{a^3} ds.$$

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Groups and unitary representations

A general framework for all above examples is given by

$$f = \int_G \langle f, \mu_g \eta \rangle_{L^2(\mathbb{R}^d)} \mu_g \eta \, dg, \qquad f \in L^2(\mathbb{R}^d), \tag{1}$$

(in the weak sense), where

- G is a Lie group with left Haar measure dg;
- $\mu \colon G \to \mathcal{U}(L^2(\mathbb{R}^d))$ is a unitary representation;
- η is called mother wavelet or *admissible vector;*
- the map

$$V_\eta \colon L^2(\mathbb{R}^d) \to L^2(G), \qquad V_\eta f(g) = \langle f, \mu_g \eta \rangle_{L^2(\mathbb{R}^d)}$$

  is called *voice transform.*

The building blocks are obtained by an action of G on η.

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The group G and the representation μ are called *reproducing* if (1) holds weakly for every $f \in L^2(\mathbb{R}^d)$.

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Wavelets

One of the most relevant examples of reproducing formula

$$f = \int_G \langle f, \mu_g \eta \rangle_{L^2(\mathbb{R}^d)} \mu_g \eta \, dg, \qquad f \in L^2(\mathbb{R}^d),$$

is given by the wavelet analysis.

1. One-dimensional wavelets: $d = 1$, $G = \mathbb{R} \rtimes \mathbb{R}_+$ and

$$\mu_{(b,a)}\eta = a^{-1/2}\eta((x-b)/a), \qquad \eta \in L^2(\mathbb{R}).$$

2. Multi-dimensional wavelets: $G = \mathbb{R}^d \rtimes \mathbb{R}_+$ and

$$\mu_{(b,a)}\eta = a^{-d/2}\eta((x-b)/a), \qquad \eta \in L^2(\mathbb{R}^d).$$

3. tensor products of 1D wavelets: $G = \mathbb{R}^d \rtimes \mathbb{R}^d_+$, $\mu_{(b_i,a_i)}\eta \approx \eta((x_i - b_i)/a_i)$
4. directional wavelets in 2D: $G = \mathbb{R}^2 \rtimes (\mathbb{R}_+ \times S^1)$ and

$$\mu_{(b,a,\theta)}\eta = a^{-1}\eta(R_\theta(x-b)/a), , \qquad \eta \in L^2(\mathbb{R}^2),$$

where R_θ is the rotation matrix.

Characterization of admissible vectors: in 1. we need

$$\int_{\mathbb{R}_+} \frac{|\hat{\eta}(\pm\xi)|^2}{\xi} d\xi = 1.$$

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In recent years, many variations of the wavelets have been considered: ridgelets (Candes and Donoho, 1999), curvelets (Candes and Donoho, 2004), contourlets (Do, Vetterli, 2005), shearlets (Labate, Lim, Kutyniok, Weiss, 2005).

Let us focus on one of these generalisations which is based on a group action: the shearlets.

- $G = TDS(2) = \mathbb{R}^2 \rtimes (\mathbb{R}_+ \times \mathbb{R})$: translations b, dilations a and shearings s.
- Continuous shearlet transform given by

$$\mu_{(b,a,s)}\eta(x) = a^{-3/4}\eta\left(S_s^{-1}D_a^{-1}(x-b)\right), \qquad \eta \in L^2(\mathbb{R}^2),$$

  where
  - $S_s = \left[\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right]$ is the shearing matrix
  - $D_a = \left[\begin{smallmatrix} a & 0 \\ 0 & a^{1/2} \end{smallmatrix}\right]$ is the (anisotropic) dilation matrix.
- Admissibility condition:

$$\int_{\mathbb{R}_+\times\mathbb{R}} \frac{|\hat{\eta}(\pm\xi_1,\xi_2)|^2}{\xi_1^2}\, d\xi_1 d\xi_2.$$

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A unified approach to signal analyses

- The above reproducing systems were introduced and studied separately, and the characterization of the admissible vectors achieved with ad-hoc approaches.
- Is a unified approach to signal analyses possible?
- It has been observed in (Cordero, De Mari, Nowak, Tabacco, 2006, 2006, 2010) that many of the above representations (including wavelets, tensor product of wavelets, directional wavelets and shearlets) were equivalent to the restriction of the metaplectic representation to a particular class of subgroups of the symplectic group.
- Some of these were studied in (Cordero, Tabacco, 2013).
- In this talk:
  1. Classification, up to conjugation, of all the groups in this class in 2D (A, Balletti, De Mari, De Vito, 2013)
  2. Classification, up to conjugation and orbit equivalence, of all the reproducing groups + admissible vectors (A, De Mari, De Vito, Mantovani, 2014)
  3. Discretisation of the Schrödingerlets (A, Dahlke, De Mari, De Vito, Vigogna, in preparation)

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A unified approach to signal analyses

- The above reproducing systems were introduced and studied separately, and the characterization of the admissible vectors achieved with ad-hoc approaches.
- Is a unified approach to signal analyses possible?
- It has been observed in (Cordero, De Mari, Nowak, Tabacco, 2006, 2006, 2010) that many of the above representations (including wavelets, tensor product of wavelets, directional wavelets and shearlets) were equivalent to the restriction of the metaplectic representation to a particular class of subgroups of the symplectic group.
- Some of these were studied in (Cordero, Tabacco, 2013).
- In this talk:
  1. Classification, up to conjugation, of all the groups in this class in 2D (A, Balletti, De Mari, De Vito, 2013)
  2. Classification, up to conjugation and orbit equivalence, of all the reproducing groups + admissible vectors (A, De Mari, De Vito, Mantovani, 2014)
  3. Discretisation of the Schrödingerlets (A, Dahlke, De Mari, De Vito, Vigogna, in preparation)

Outline of the talk

A particular class of groups

The groups we are interested in are triangular subgroups of $Sp(d,\mathbb{R})$ of the form

$$G = \Sigma \rtimes H = \{g_{(\sigma,h)} = \begin{bmatrix} h & 0 \\ \sigma h & {}^t h^{-1} \end{bmatrix} : h \in H, \sigma \in \Sigma\} \subset Sp(d,\mathbb{R}),$$

where

- Σ is a non-trivial vector space of $Sym(d,\mathbb{R})$ (symmetric matrices);
- H is a non-trivial connected subgroup of $GL(d,\mathbb{R})$ contained in

$$H(\Sigma) = \{h \in GL(d,\mathbb{R}) : {}^t h^{-1} \sigma h^{-1} \in \Sigma \text{ for all } \sigma \in \Sigma\};$$

- the group law is $g_{(\sigma_1,h_1)} g_{(\sigma_2,h_2)} = g_{(\sigma_1 + {}^t h_1^{-1} \sigma_2 h_1^{-1}, h_1 h_2)}$.

In this case we say that $\Sigma \rtimes H$ belongs to the class $\mathcal{E}$. The metaplectic representation restricted to $\Sigma \rtimes H$ takes the form

$$\mu_{g(\sigma,h)} f(\xi) = |\det h|^{-1/2} e^{\pi i \langle \sigma\xi, \xi\rangle} f(h^{-1}\xi)$$

- $f(h^{-1}\xi)$: geometric transformations
- $e^{\pi i \langle \sigma\xi, \xi\rangle}$: translations (frequency modulations)

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Wavelets and shearlets in the class $\mathcal{E}$

- $\Sigma = \mathbb{R}\,, \quad H = \mathbb{R}_+\,, \quad G = \mathbb{R} \rtimes \mathbb{R}_+ \in \mathcal{E}$
- "$ax+b$"$\ni (b,a) \overset{\zeta}{\mapsto} g(b, a^{-1/2}) \in G$ is a group isomorphism

The metaplectic representation

$$\mu_{\zeta(b,a)}\, f(\xi) = a^{\frac{1}{4}}\, e^{\pi i b \xi^2}\, f\left(a^{\frac{1}{2}}\xi\right)$$

The wavelet representation

$$W'_{(b,a)} f(\xi) = \sqrt{a}\, e^{-2\pi i \xi b}\, f(a\,\xi)$$

There exists $\Psi\colon L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_-) \to L^2(\mathbb{R})$ such that

$$\begin{array}{ccc} L^2(\mathbb{R}) & \xrightarrow{\mu_{\zeta(b,a)}} & L^2(\mathbb{R}) \\ \uparrow \Psi & & \downarrow \Psi^{-1} \\ L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_-) & \xrightarrow{U'_{(b,a)} \oplus U'_{(b,a)}} & L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_-) \end{array}$$

Similarly, multi-dimensional wavelets and shearlets are equivalent, in the sense precised above, to the metaplectic representations restricted to other groups in $\mathcal{E}$.

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The 2D classification of the groups in $\mathcal{E}$ up to conjugation

- The classification of all the groups in 2D of the form

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should be done up to conjugation by elements of $Sp(2,\mathbb{R})$, since conjugated groups give rise to equivalent representations.

- $\Sigma \rtimes H \in \mathcal{E}$ iff $\Sigma^\perp \rtimes {}^t H \in \mathcal{E}$: consider only $\dim \Sigma = 1$:

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- The simple conjugations

$$g(0,q)(\Sigma \rtimes H)g(0,q)^{-1} = ({}^t q^{-1} \Sigma q^{-1}) \rtimes (qHq^{-1})$$

preserve the class E, and allow to diagonalise the generator of Σ.
- Explicitly compute $H(\Sigma)$ and all its connected subgroups (via subalgebras)
- Conjugations by general elements of $Sp(2,\mathbb{R})$ much more involved.
- Complete classification in (A, Balletti, De Mari, De Vito, 2013)

Important task: selecting the reproducing groups in $\mathcal{E}$

- The theory in (De Mari, De Vito, 2013) for irreducible representations allows to determine whether these groups are reproducing and to characterise the a.v.

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$$g(0,q)(\Sigma \rtimes H)g(0,q)^{-1} = ({}^tq^{-1}\Sigma q^{-1}) \rtimes (qHq^{-1})$$

  preserve the class E, and allow to diagonalise the generator of Σ.
- Explicitly compute $H(\Sigma)$ and all its connected subgroups (via subalgebras)
- Conjugations by general elements of $Sp(2,\mathbb{R})$ much more involved.
- Complete classification in (A, Balletti, De Mari, De Vito, 2013)

Important task: selecting the reproducing groups in $\mathcal{E}$

- The theory in (De Mari, De Vito, 2013) for irreducible representations allows to determine whether these groups are reproducing and to characterise the a.v.

The 2D classification of the groups in $\mathcal{E}$ up to conjugation

- The classification of all the groups in 2D of the form

$$G = \Sigma \rtimes H = \{g_{(\sigma,h)} = \begin{bmatrix} h & 0 \\ \sigma h & {}^t h^{-1} \end{bmatrix} : h \in H, \sigma \in \Sigma\} \subset Sp(2,\mathbb{R}),$$

  should be done up to conjugation by elements of $Sp(2,\mathbb{R})$, since conjugated groups give rise to equivalent representations.
- $\Sigma \rtimes H \in \mathcal{E}$ iff $\Sigma^\perp \rtimes {}^t H \in \mathcal{E}$: consider only $\dim \Sigma = 1$:

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The reproducing groups in $\mathcal{E}$ in 2D (A, De Mari, De Vito, Mantovani, 2014)

(4D.1) $\Sigma_3 \rtimes \{\begin{bmatrix} a_1 & 0 \\ s & a_2 \end{bmatrix} : s \in \mathbb{R}, a_1, a_2 > 0\}$

(4D.2) $\Sigma_1^\perp \rtimes \{aR_s : s \in \mathbb{R}, a > 0\}$

(4D.3) $\Sigma_2^\perp \rtimes \{aA_s : s \in \mathbb{R}, a > 0\}$

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The Schrödingerlets (Dahlke, De Mari, De Vito, Labate, Steidl, Teschke, Vigogna, 2014)

- As an example, we consider the newly introduced reproducing system given by

$$\Sigma_1 \rtimes \{aR_\theta\} = \{\left[\begin{smallmatrix} b & 0 \\ 0 & b \end{smallmatrix}\right] : b \in \mathbb{R}\} \rtimes \{a \left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right] : \theta \in S^1, a > 0\},$$

namely with 1D translations, dilations and rotations (not necessary).
- Setting $\hat{\mu}_b = \mathcal{F}^{-1}\mu(b,1,0)\mathcal{F}$, there holds

$$(2\pi i \frac{\partial}{\partial b} + \Delta)\hat{\mu}_b f(x) = 0,$$

namely $\hat{\mu}_b f$ solves the Schrödinger equation with b as a time variable.
- Write $L^2(\mathbb{R}^2)$ in polar coordinates $L^2(\mathbb{R} \times S^1) = L^2(\mathbb{R}) \otimes L^2(S^1)$ and for simplicity consider only $f(x,\varphi) \in \mathcal{F}_x^{-1} L^2(\hat{\mathbb{R}}_+) \otimes L^2(S^1)$.
- The representation is equivalent to

$$\pi(b,a,\theta)f(x,\varphi) = a^{-1/2} f((x-b)/a, \varphi - \theta), \quad \pi = w \otimes \lambda = \oplus_n w \otimes e_{-n}.$$

- A function $\eta(x,\varphi) = \sum_n \eta_n(x) e^{in\varphi}$ is an admissible vector iff

$$\int_{\mathbb{R}_+} |\mathcal{F}_x \eta_n(\xi)|^2 \frac{d\xi}{\xi} = 1, \qquad n \in \mathbb{Z}.$$

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Outline of the talk

Discretisation and coorbit theory

$$f = \int_G \langle f, \mu_g \eta \rangle_{L^2(\mathbb{R}^d)} \mu_g \eta \, dg \quad \xrightarrow{?} \quad f = \sum_i \langle f, \mu_{g_i} \eta \rangle_{L^2(\mathbb{R}^d)} \mu_{g_i} \eta$$

- The classical way of discretising reproducing formulae uses the coorbit theory (Feichtinger, Grochenig, 1986, 1989), and has been applied in many cases (wavelets, shearlets...)
- However, the coorbit theory is valid for integrable representation, i.e. with integrable kernel
$$K := V_\eta \eta = \langle \eta, \mu_g \eta \rangle_{L^2(\mathbb{R}^d)} \in L^1$$
- Hard to find L^1 kernels in our cases, e.g. with the Schrödingerlets (Dahlke et al., 2014), because they are series of irreducible components.
- A coorbit theory for non-integrable representations has been recently considered (Dahlke et al., 2014), but the discretisation issue has not been incorporated yet.
- The main problem in the case $K \notin L^1$ is the generalisation of
$$\|K * F\|_p \leq \|K\|_1 \|F\|_p ,$$
because of the loss of integrability given by Young's inequality.

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Definition

A subset $\{\psi_i\}_{i \in I}$ of a Hilbert space $\mathcal{H}$ is called a *frame* if there exist $A, B > 0$ such that

$$A \|f\|_{\mathcal{H}}^2 \leq \sum_{i \in I} |\langle f, \psi_i \rangle|^2 \leq B \|f\|_{\mathcal{H}}^2, \qquad f \in \mathcal{H}.$$

If $A = B = 1$ then $\{\psi_i\}_i$ is called Parseval frame.

- Note that if $\{\psi_i\}_i$ is a Parseval frame then we have the discrete representation

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The discretisation of the Schrödingerlets

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- $G = \mathbb{R} \rtimes (\mathbb{R}_+ \times S^1)$
- $\pi_{(b,a,\theta)} f(x, \varphi) = a^{-1/2} f((x-b)/a, \varphi - \theta), \quad f \in \mathcal{F}_x^{-1} L^2(\hat{\mathbb{R}}_+) \otimes L^2(S^1).$
- $\eta(x, \varphi) = \sum_n \eta_n(x) e^{in\varphi}$, $\eta_n(x) = \alpha_n \eta_0(\alpha_n x)$.
- Take the sampling of G for some $L \in \mathbb{N}^*$:

$$g_{k,j,l} = (2^j k, 2^j, 2\pi l/L) : (k, j, l) \in \mathbb{Z}^2 \times \{0, \ldots, L-1\}$$

Theorem (A, Dahlke, De Mari, De Vito, Vigogna, 2015)

Assume that $\operatorname{supp} \mathcal{F}_x \eta_0 \subseteq [0, 1]$ *and fix* $L \in \mathbb{N}^*$. *There exist* $\alpha_n > 0$ *such that*

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Take $\mathcal{H} = \oplus_{n\in\mathbb{Z}} \mathcal{H}_n$ and let $P_n \colon \mathcal{H} \to \mathcal{H}_n$ be the canonical projections. A family $\{\psi_i\}_{i\in I}$ is a Parseval frame for $\mathcal{H}$ if and only if

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- Unified approach by considering the metaplectic representation restricted to a particular class of subgroups of the symplectic group.
- Full classification in the 2D case: old and new formulae.
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- Use the PDE for the discretisation of the Schrödingerlets.
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