Disjoint sparsity for signal separation and applications to hybrid imaging inverse problems

Giovanni S Alberti

Seminar for Applied Mathematics, ETH Zurich

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ETH zürich

Outline

1. Introduction to hybrid imaging inverse problems
2. Disjoint sparsity for signal separation
3. Applications to quantitative photoacoustic tomography

Giovanni S. Alberti and Habib Ammari.
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Applied and Computational Harmonic Analysis, 2015.
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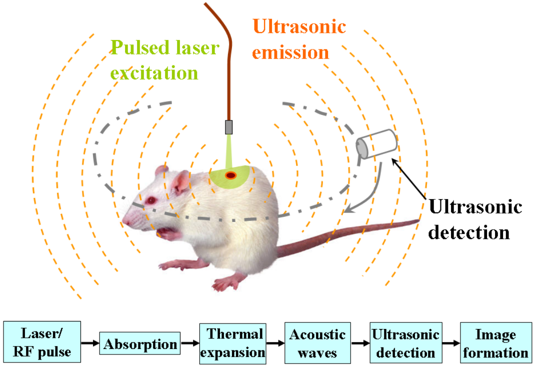
Photoacoustic tomography



(From Wikipedia, http://en.wikipedia.org/wiki/Photoacoustic_imaging_in_biomedicine)

1. The image is $H(x) = \Gamma(x)\mu(x)u(x)$, where
   - μ is the light absorption,
   - Γ is the Grüneisen parameter,
   - and u is the light intensity.
2. How to extract the unknowns from H?

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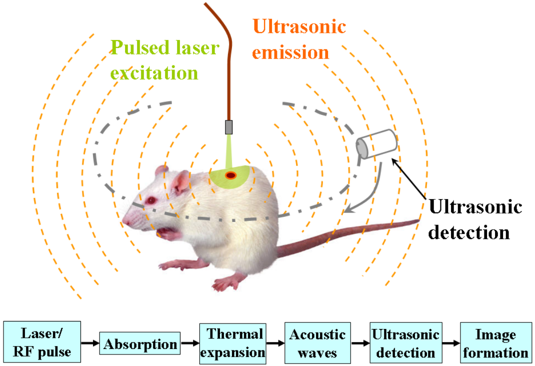

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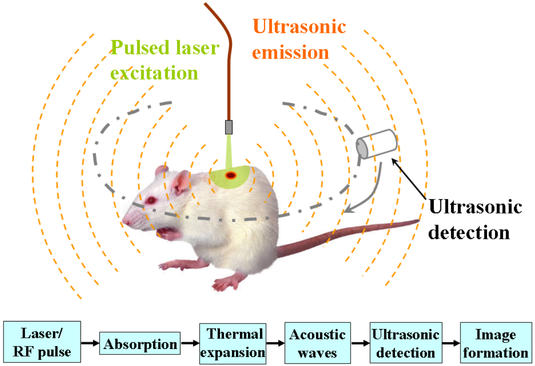

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PDE-based methods

- Alessandrini, Arridge, Bal, Beard, Beretta, Cox, Di Cristo, Francini, Gao, Jollivet, Jugnon, Kaipio, Köstli, Laufer, Muszkieta, Naetar, Pulkkinen, Ren, Scherzer, Tarvainen, Uhlmann, Vessella, Zhao, ...
- A possible way to obtain Γ and μ from

$$H = \Gamma \mu u$$

is based on the PDE satisfied by u. In the diffusive regime for light

$$-\mathrm{div}(D\nabla u) + \mu u = 0 \quad \text{in } \Omega.$$

- This approach is sometimes very successful. Possible drawbacks:
  - PDE model non accurate (e.g. transport regime for light), or required boundary conditions not known.
  - Too many unknowns (e.g. if $\Gamma \neq 1$ above)

The focus of this talk is a new approach to this issue based on the separation of the unknowns from the fields via sparsity conditions:

$$h = \log H = \log \Gamma \mu + \log u = f + g.$$

- Following [Rosenthal, Razansky, Ntziachristos, 2009], observe that:
  - the light absorption μ is a constitutive parameter of the tissue, and as such is discontinuous. Its discontinuities are typically the inclusions we are looking for;
  - the light intensity u tends to be much smoother.

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Introduction to sparse representations

- Let $h \in \mathbb{R}^n$ be a column vector ($n = d \times d$ is the resolution of the image).
- Let $A \in \mathbb{R}^{n \times m}$ be a dictionary of m atoms, which are used as building blocks:

$$h = Ay, \tag{1}$$

  for some coefficient vector $y \in \mathbb{R}^m$ (weights).
- If $m > n$ then (1) is in general underdetermined, and has many solutions y.
- Select the *sparsest* one, i.e. with fewest non-zero entries:

$$\min_{y \in \mathbb{R}^m} \|y\|_0 \quad \text{subject to } h = Ay,$$

  where $\|y\|_0 := \#\mathrm{supp}\, y = \#\{\alpha \in \{1, \dots, m\} : y(\alpha) \neq 0\}$.
- If the dictionary A is well chosen, it is possible to represent an n-dimensional vector with much fewer coefficients.
- In practice, we minimise

$$\min_{y \in \mathbb{R}^m} \|y\|_0 \quad \text{subject to } \|h - Ay\|_2 \leq \varepsilon$$

  for some $\varepsilon > 0$ (hard problem, but many available algorithms).

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Morphological component analysis (Starck, Elad, Donoho, 2004,...)

Back to the signal separation problem.

- Let $h = f + g \in \mathbb{R}^n$ be the sum of two components.
- Let $A_f \in \mathbb{R}^{n \times m_f}$ and $A_g \in \mathbb{R}^{n \times m_g}$ be two dictionaries such that:
  - f can be sparsely represented w.r.t. A_f but not w.r.t. A_g;
  - g can be sparsely represented w.r.t. A_g but not w.r.t. A_f;
- Decompose h w.r.t. the concatenated dictionary $A = [A_f, A_g]$:

$$\min_{y \in \mathbb{R}^{m_f + m_g}} \|y\|_0 \quad \text{subject to } [A_f, A_g] \begin{bmatrix} y_f \\ y_g \end{bmatrix} = h.$$

- Recover $f \approx A_f y_f$, $g \approx A_g y_g$.

In other words, the original components can be recovered provided that they have different features. This is expressed in terms of the sparsity of their decompositions with different dictionaries.

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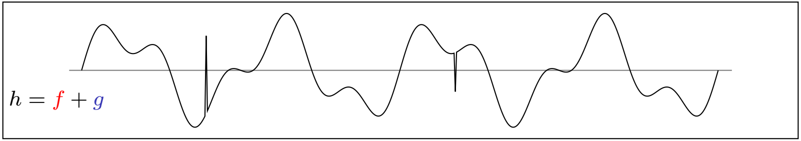
Example: spikes and sinusoids (Donoho, Huo, 2001,...)



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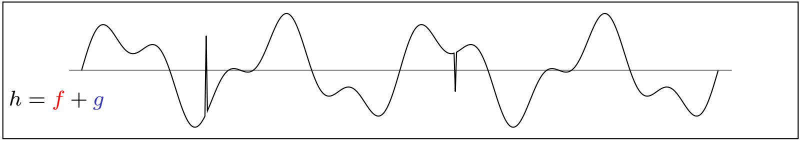

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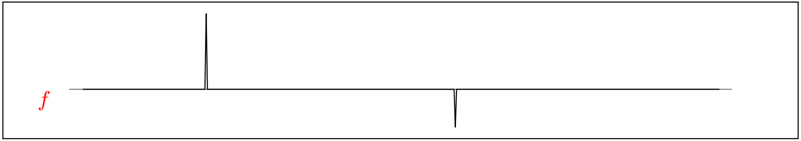

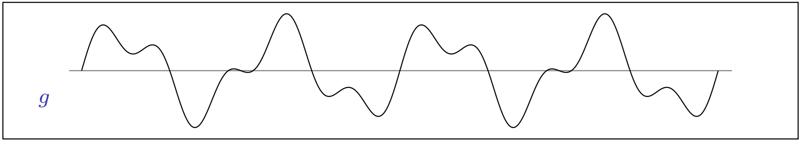

Theoretical justification (Elad & Bruckstein, 2002)

Why does writing $f = A_f y_f$ and $g = A_g y_g$ give the correct reconstruction?

- (Uncertainty principle) If $h \neq 0$ has the representations $h = Ay_A = By_B$ w. r. t. two orthonormal bases $A = [a_1, \dots, a_n]$ and $B = [b_1, \dots, b_n]$, then

  $$\|y_A\|_0 + \|y_B\|_0 \geq 2/M,$$

  where $M = \max_{i,j} |(a_i, b_j)_2|$ is the *mutual coherence*. ($M = 1/\sqrt{n}$ with spikes and sinusoids.)
- If f and g have representations y_f and y_g satisfying $\|y_f\|_0 + \|y_g\|_0 < 1/M$, then the reconstruction is correct.
- In practice, the assumption $\|y_f\|_0 + \|y_g\|_0 < 1/M$ is almost never satisfied, and so the above argument remains only a theoretical speculation.

Focus of this new approach: extension to multiple measurements (many other generalisations by Georgiev, Theis, Cichocki, Bobin, Moudden, Starck, Donoho, Kutyniok...).

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Focus of this new approach: extension to multiple measurements (many other generalisations by Georgiev, Theis, Cichocki, Bobin, Moudden, Starck, Donoho, Kutyniok...).

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Why does writing $f = A_f y_f$ and $g = A_g y_g$ give the correct reconstruction?

- (Uncertainty principle) If $h \neq 0$ has the representations $h = Ay_A = By_B$ w. r. t. two orthonormal bases $A = [a_1, \ldots, a_n]$ and $B = [b_1, \ldots, b_n]$, then

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Multi-measurement case

- Let
$$h_i = f + g_i \in \mathbb{R}^n, \qquad i = 1, \dots, N$$
be N measurements. The problem is to recover f and the g_i's.
- Let $A_f \in \mathbb{R}^{n \times m_f}$ and $A_g \in \mathbb{R}^{n \times m_g}$ be two dictionaries as before. Assume that A_g is an orthonormal set (and that A_f is an orthonormal basis). In our applications:
  - $A_f =$ Haar wavelets,
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- The reconstruction method applied here consists in the minimisation of
$$\min_{y \in \mathbb{R}^{m_f + N m_g}} \|y\|_0 \quad \text{subject to } \left\| [A_f, A_g] \begin{bmatrix} y_f \\ y_g^i \end{bmatrix} - h_i \right\|_2 \leq \varepsilon, \ i = 1, \dots, N,$$
where $y = {}^t[{}^t y_f, {}^t y_g^1, \dots, {}^t y_g^N]$.
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- "Haar wavelets". Let A_f be the orthobasis of 2D Haar wavelets in $\mathbb{R}^{2^7 \times 2^7}$ and let A_g be 960 low frequency non-constant sinusoids. There exists $D > 0$ s.t.

$$\|{}^tA_f p\|_0 \geq 1160,$$

for every $p \in \mathbb{R}^n$ such that $\|p\|_2 > D$ and $\|{}^tA_g^{\perp} p\|_2 \leq 2/3$.

Uncertainty principles

2. ... for every $p \in \mathbb{R}^n$ such that $\|p\|_2 > D$ and $\|{}^tA_g^{\perp} p\|_2 \leq 2/3$ there holds

$$\#(\operatorname{supp} {}^tA_f p \setminus \operatorname{supp} \tilde{y}_f) + \sum_{i=1}^{N} \# \left(\{\alpha : |({}^tA_g p)(\alpha)| \geq 1\} \setminus \operatorname{supp} \tilde{y}_g^i\right) > m_g + \|\tilde{y}_f\|_0.$$

- "Classical". If M is the mutual coherence of A_f and A_g:

$$\|{}^tA_f p\|_0 + \|{}^tA_g p\|_0 \geq 2/M.$$

- "Normalised". $\exists D > 0$ s.t. for all $p \in \mathbb{R}^n$ with $\|p\|_2 > D$ and $\|{}^tA_g^{\perp} p\|_2 \leq 2/3$

$$\|{}^tA_f p\|_0 + \#\{\alpha : |({}^tA_g p)(\alpha)| \geq 1\} \gtrsim 2.$$

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Main result

The following result states that the separation method with multiple measurements gives unique and stable reconstruction.

Theorem

Assume 1 and 2 and that $\varepsilon := \rho_f + \rho_g + \eta \leq \beta/3$. *Assume that* $f, g_i, n_i \in \mathbb{R}^n$ *satisfy* $\|n_i\|_2 \leq \eta$ *and*

$$\left\|A_f \tilde{y}_f - f\right\|_2 \leq \rho_f, \quad \left\|A_g \tilde{y}_g^i - g_i\right\|_2 \leq \rho_g, \qquad i = 1, \ldots, N,$$

and let $y_f \in \mathbb{R}^{m_f}$ *and* $y_g^i \in \mathbb{R}^{m_g}$ *realise the minimum of*

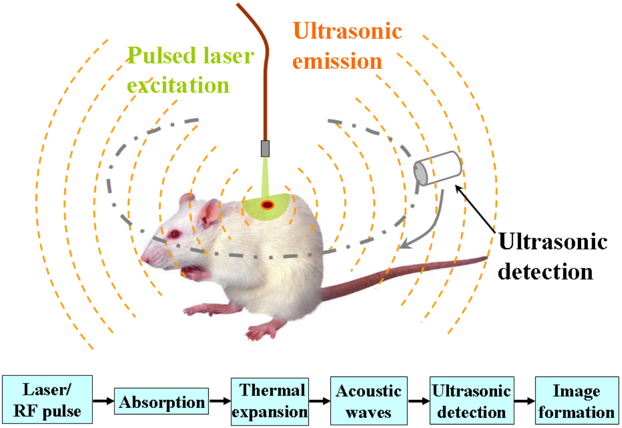
$$\min_{y \in \mathbb{R}^{m_f + N m_g}} \|y\|_0 \quad \text{subject to } \left\|[A_f, A_g]\begin{bmatrix} y_f \\ y_g^i \end{bmatrix} - h_i\right\|_2 \leq \varepsilon,\ i = 1, \ldots, N,$$

where $h_i = f + g_i + n_i$. *Then*

$$\left\|A_f y_f - f\right\|_2 \leq (3D+1)\varepsilon, \quad \left\|A_g y_g^i - g_i\right\|_2 \leq (3D+2)\varepsilon, \qquad i = 1, \ldots, N.$$

Outline of the talk

Quantitative photoacoustic tomography, $\Gamma = 1$



It gives us the image $H(x) = \Gamma(x)\mu(x)u(x) = \mu(x)u(x)$. Need to find μ.

In the general case $\Gamma \neq 1$, apply the same approach and find $\Gamma\mu$ and u. Then by using the PDE approach all the unknowns can be reconstructed.

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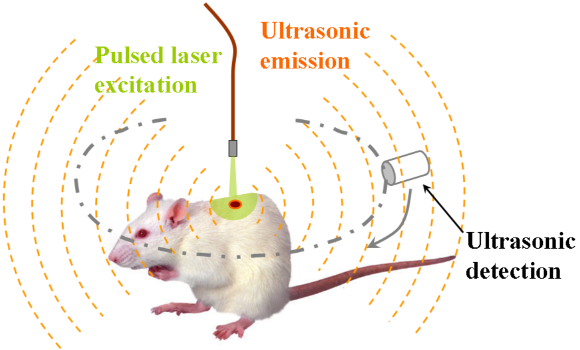

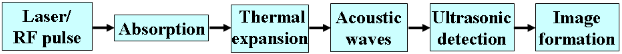

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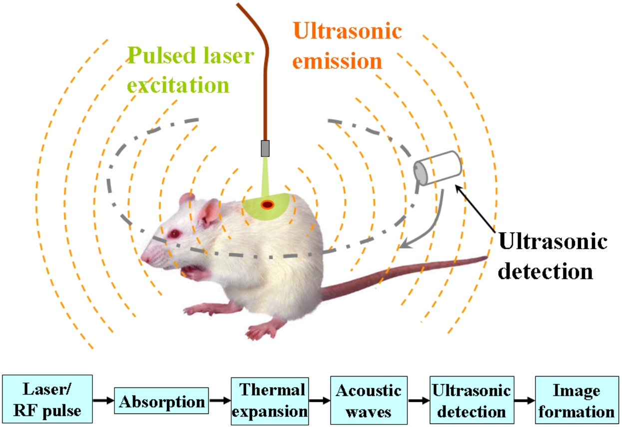

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Quantitative photoacoustic tomography, $\Gamma = 1$

- Multi-measurement data: $H(x) = \mu(x)u_i(x)$, where

$$\begin{cases} -\Delta u_i + \mu u_i = 0 & \text{in } \Omega, \\ u_i = \varphi_i & \text{on } \partial\Omega. \end{cases}$$

- Taking a log: $h_i = \log \mu + \log u_i$.
- The above method may be used if $\log \mu$ and $\log u_i$ can be sparsely represented with respect to two different dictionaries.
- Following [Rosenthal, Razansky, Ntziachristos, 2009], observe that:
  - the light absorption μ is a constitutive parameter of the tissue, and as such is discontinuous. Its discontinuities are typically the inclusions we are looking for;
  - the light intensity u_i is a solution of a PDE, and as such enjoys higher regularity properties.
- Thus, the uncertainty principle motivates these for the dictionaries:
  - A_f: Haar wavelets (curvelets, ridgelets, shearlets...)
  - A_g: low frequency trigonometric polynomials (constant included!)
- The disjoint sparsity signal separation method may be used to recover μ.

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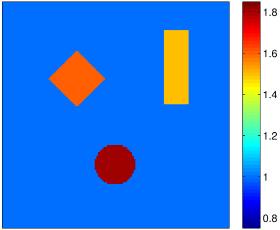
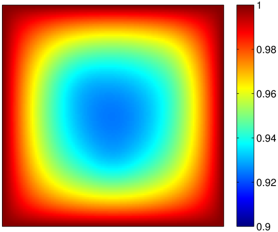
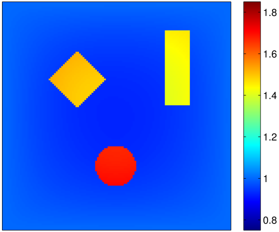
Example 1: noise-free case, $N = 1$



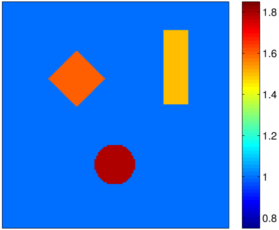
(a) The true absorption $\tilde{\mu}$



(b) The true intensity $\tilde{u}_1$



(c) The datum H_1



(d) The reconstructed μ

Video: $N = 1$. Video: $N = 2$.

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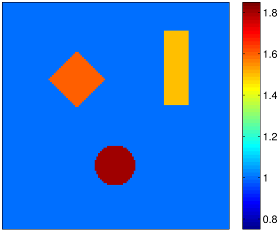

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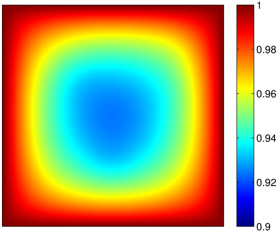

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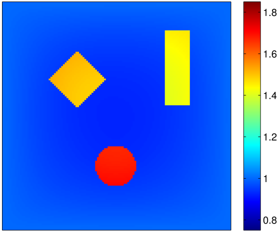

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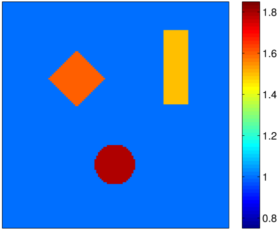
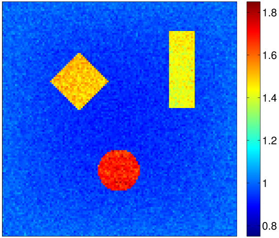


(d) The reconstructed μ

Video: $N = 1$. Video: $N = 2$.

Example 1: noisy case, $N = 1, 3, 5$



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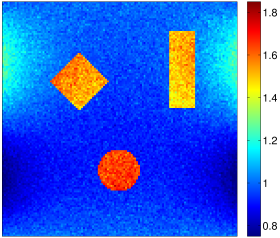
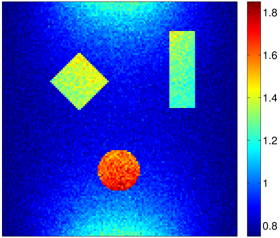
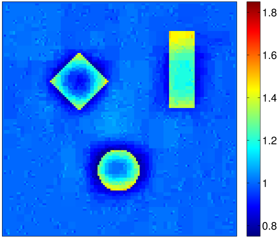


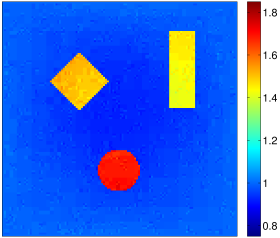
(b) The datum H_3



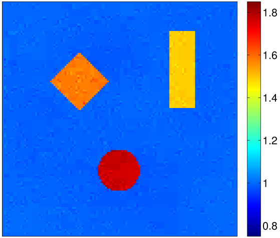
(c) The datum H_5



(d) Reconstructed μ, $N = 1$

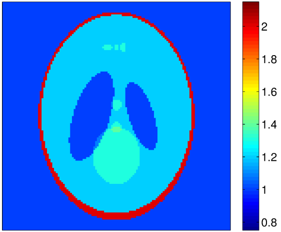


(e) Reconstructed μ, $N = 3$

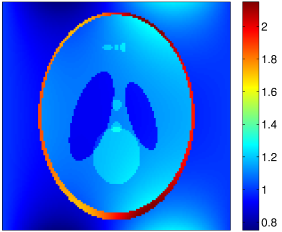


(f) Reconstructed μ, $N = 5$

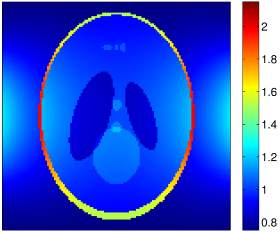
Example 2: the Shepp-Logan Phantom



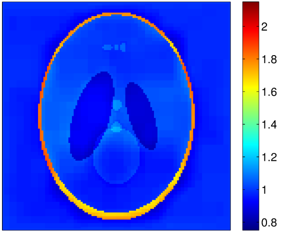
(a) $\tilde{\mu}$



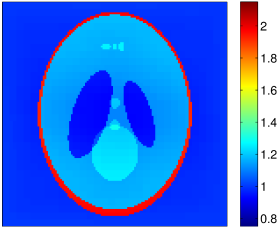
(b) H_2



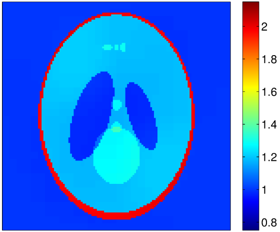
(c) H_4



(d) μ, $N = 1$

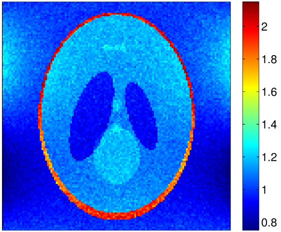


(e) μ, $N = 3$

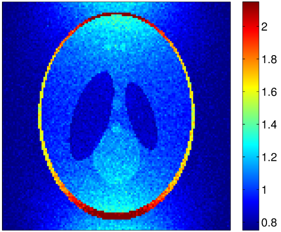


(f) μ, $N = 5$

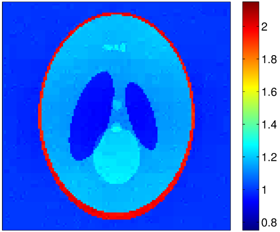
Example 2: the Shepp-Logan Phantom with noise



(a) H_3

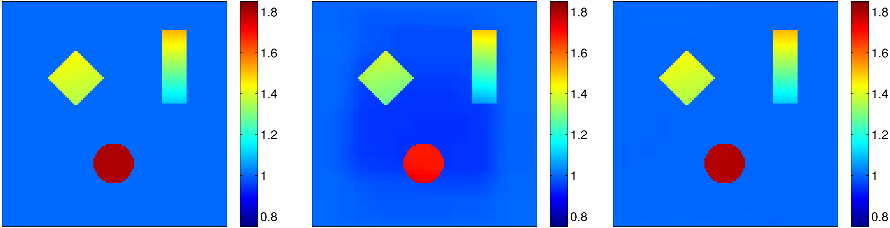


(b) H_5



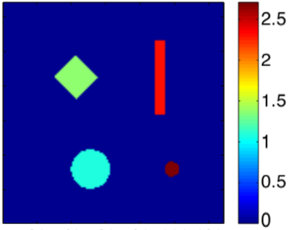
(c) μ, $N = 5$

Example 3: piecewise smooth

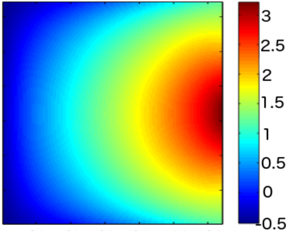


(a) $\tilde{\mu}$ (b) $\mu,\ N = 1$ (c) $\mu,\ N = 2$

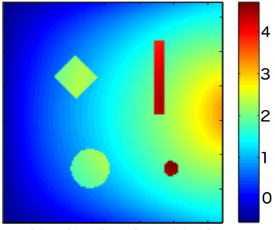
Example 4: lateral illuminations, $N = 4$



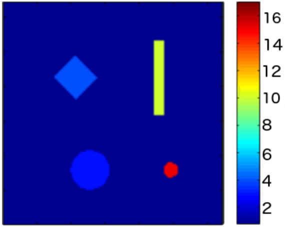
(a) $\log \tilde{\mu}$



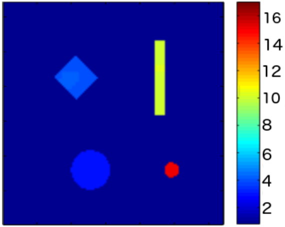
(b) $\log \tilde{u}_1$



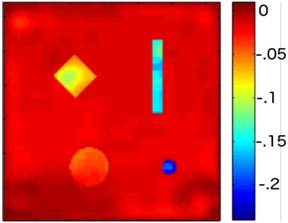
(c) $\log H_1$



(d) $\tilde{\mu}$



(e) μ



(f) $\|\mu - \tilde{\mu}\|_2 \approx 1.5 \cdot 10^{-2}$

Conclusions

Past

- The reconstruction in QPAT (and in many hybrid imaging inverse problems) require the separation of several signals $h_i = f + g_i$.
- PDE techniques are often powerful, but sometimes they are not applicable.

Present

- Multiple measurements and disjoint sparsity can be used to find f and g_i.
- Uniqueness and stability proven.
- Orthogonal matching pursuit performs well in many numerical simulations related to quantitative photoacoustic tomography.

Future

- How can we ensure that the light intensities u_i give the necessary incoherence, measured in terms of their disjoint sparsity? Random illuminations may help.
- Norm $l^0 \rightarrow$ norm l^1.
- Robust uncertainty principles (Candes, Romberg, 2006)

Conclusions

Past

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- PDE techniques are often powerful, but sometimes they are not applicable.

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- Uniqueness and stability proven.
- Orthogonal matching pursuit performs well in many numerical simulations related to quantitative photoacoustic tomography.

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