Non-zero constraints in quantitative coupled physics imaging

Giovanni S. Alberti

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Quantitative Tomographic Imaging – Radon meets Bell and Maxwell

Internal data in quantitative hybrid imaging problems

- Hybrid conductivity imaging [Widlak, Scherzer, 2012]

$$\begin{cases} -\mathrm{div}(a\,\nabla u^i) = 0 & \text{in } \Omega, \\ u^i = \varphi_i & \text{on } \partial\Omega. \end{cases}$$

$$u^i(x) \quad \text{or} \quad a(x)\,\nabla u^i(x) \quad \text{or} \quad a(x)\,\left|\nabla u^i\right|^2(x) \qquad \xrightarrow{?} \qquad a$$

- Quantitative thermoacoustic tomography [Bal et al., 2011, Ammari et al., 2013]

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$$\sigma(x)\,\left|u^i\right|^2(x) \qquad \xrightarrow{?} \qquad \sigma$$

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$$H^i(x) \qquad \xrightarrow{?} \qquad \varepsilon, \sigma$$

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Why do non-zero constraints matter?

- Consider for simplicity the hybrid conductivity problem

$$\begin{cases} -\operatorname{div}(a\,\nabla u) = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

with internal data ∇u and unknown a.

- With 1 measurement:

$$\nabla a \cdot \nabla u = -a\Delta u \quad \Longrightarrow \quad \nabla(\log a)\cdot\nabla u = -\Delta u$$

This equation may be solved in a if a is known on $\partial\Omega$ and if

$$\nabla u(x) \neq 0, \qquad x \in \Omega.$$

- With d measurements:

$$\nabla(\log a)\cdot(\nabla u^1,\cdots,\nabla u^d) = -(\Delta u^1,\ldots,\Delta u^d)$$
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Main question

Is it possible to find suitable illuminations φ_i so that the corresponding solutions u^i satisfy certain non-zero constraints, such as the absence of critical points?

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Outline of the talk

1. The conductivity equation
2. The Helmholtz equation
3. The Maxwell's equations

G. S. Alberti and Y. Capdeboscq. Lectures on elliptic methods for hybrid inverse problems. Technical Report 2016-46, SAM, ETH Zürich, 2016.

Guillaume Bal. Hybrid inverse problems and internal functionals. In *Inverse problems and applications: inside out. II*, volume 60 of *Math. Sci. Res. Inst. Publ.*, pages 325–368. Cambridge Univ. Press, Cambridge, 2013.

Peter Kuchment. Mathematics of hybrid imaging: a brief review. In *The mathematical legacy of Leon Ehrenpreis*, volume 16 of *Springer Proc. Math.*, pages 183–208. Springer, Milan, 2012.

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The Radò-Kneser-Choquet theorem

Theorem (Alessandrini Magnanini 1994, Bauman et al. 2000, Alessandrini Nesi 2015)

Let $\Omega \subseteq \mathbb{R}^2$ be a $C^{1,\alpha}$ bounded convex domain and $a \in C^{0,\alpha}(\overline{\Omega}; \mathbb{R}^{2\times 2})$ be uniformly elliptic. Let $u^i \in H^1(\Omega)$ be the solutions to

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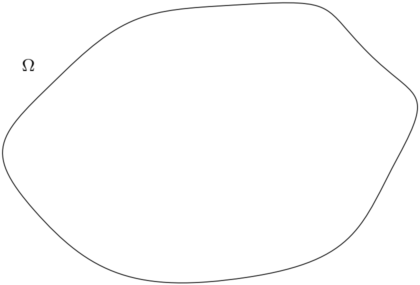

- $\det\left[\nabla u^1(x_0) \quad \nabla u^2(x_0)\right] = 0$
- Thus, $\alpha\nabla u^1(x_0) + \beta\nabla u^2(x_0) = 0$
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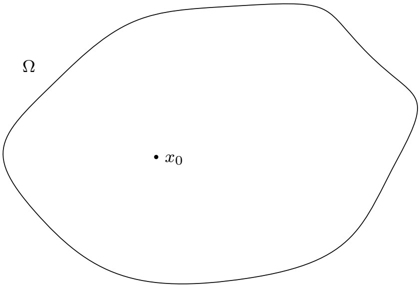

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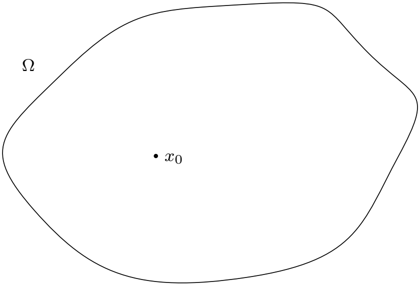

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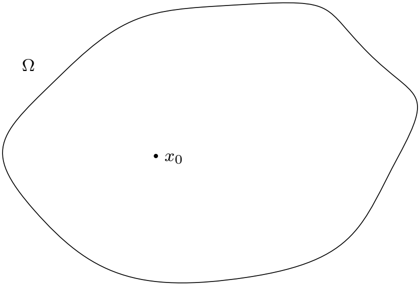

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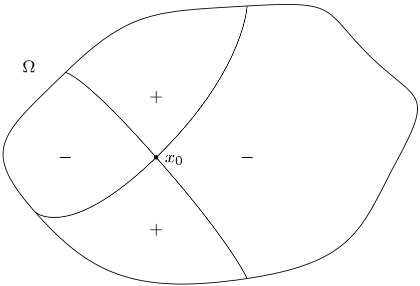

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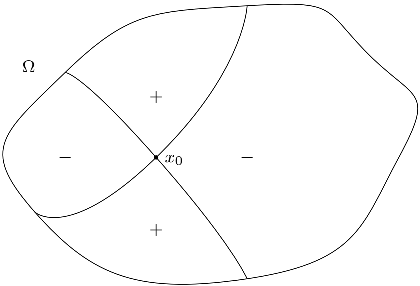

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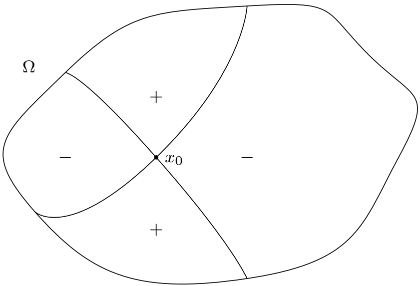

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The failure in three dimensions

$$-\mathrm{div}(a\nabla u^i) = 0 \quad \text{in } \Omega, \qquad u^i = \varphi_i \quad \text{on } \partial\Omega.$$

In three dimensions, the above result fails. Several counterexamples:

1. Laugesen 1996: the harmonic case ($a \equiv 1$) for a specific diffeomorphism $\varphi = (\varphi^1, \varphi^2)$
2. Briane et al 2004: the non-constant case (homogenization) for a specific diffeomorphism $\varphi = (\varphi^1, \varphi^2)$
3. Could it be possible to find (φ^1, φ^2) independently of a so that for every $x \in \Omega$

   $$\det \begin{bmatrix} \nabla u^1(x) & \nabla u^2(x) & \nabla u^3(x) \end{bmatrix} \neq 0?$$

   Capdeboscq 2015: No! (by using 2.)
4. What about critical points: can we find φ independently of a so that for every $x \in \Omega$

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4. What about critical points: can we find φ independently of a so that for every $x \in \Omega$

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Let $\Omega \subseteq \mathbb{R}^3$ be a bounded Lipschitz domain. Take $\varphi \in C(\partial X) \cap H^{\frac{1}{2}}(\partial X)$. There exists a (nonempty open set of) $a \in C^\infty(\overline{X})$ such that the solution $u \in H^1(X)$ to

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Can be extended to deal with:

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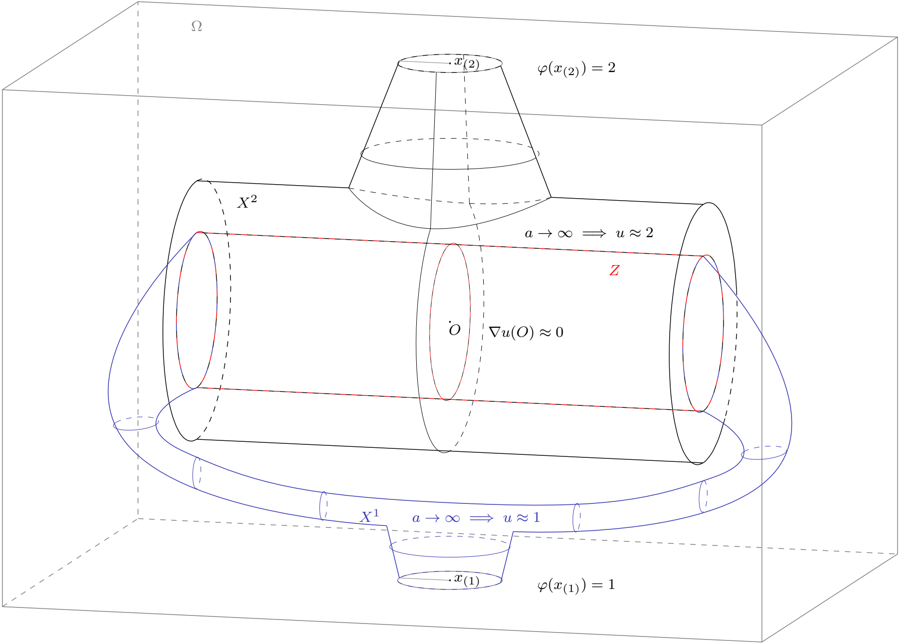

Ω
$x_{(2)}$
$\varphi(x_{(2)}) = 2$
X^2
$a \to \infty \implies u \approx 2$
Z
O
$\nabla u(O) \approx 0$
X^1
$a \to \infty \implies u \approx 1$
$x_{(1)}$
$\varphi(x_{(1)}) = 1$

Alternative approaches

- Complex geometrical optics solutions [Sylvester and Uhlmann, 1987]
  - $u^{(t)}(x) = e^{tx_m}\left(\cos(tx_l) + \mathrm{i}\sin(tx_l)\right)(1+\psi_t), \quad t \gg 1.$
  - If $t \gg 1$ then $u^{(t)}(x) \approx e^{tx_m}\left(\cos(tx_l) + \mathrm{i}\sin(tx_l)\right)$ in C^1 [Bal and Uhlmann, 2010]
  - The traces on the boundary of these solutions give the required φ_is
  - Need smooth coefficients, construction depends on coefficients.
  - Only for isotropic coefficients

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  - There exist solutions that are locally closed to the solutions of the constant coefficient PDE.
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Outline of the talk

The Helmholtz equation

- We now consider the Helmholtz equation

$$\begin{cases} \Delta u_\omega^i + (\omega^2\varepsilon + \mathrm{i}\omega\sigma)\, u_\omega^i = 0 & \text{in } \Omega, \\ u_\omega^i = \varphi_i & \text{on } \partial\Omega. \end{cases}$$

where $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$, $\varepsilon, \sigma \in L^\infty(\Omega)$, $\sigma, \varepsilon \leq \Lambda$, $\varepsilon > \Lambda^{-1}$.

- We are interested in the constraints:
  1. $\left|u_\omega^1\right|(x) > 0$ (nodal set)
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- Since solutions u_ω^i are oscillatory, they will not in general satisfy these contraints. The Radó-Kneser-Choquet theorem (constraint 2.) fails.

- CGO solutions and the Runge approximation may be used also in this case, but the corresponding boundary conditions φ_i are not explicitly constructed.

- Is there an alternative approach?

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Multi-Frequency Approach: main result

$K^{(n)}$: uniform partition of $\mathcal{A} = [K_{min}, K_{max}]$ with n points

Theorem (GSA, IP 2013 & CPDE 2015)

There exist $C > 0$ and $n \in \mathbb{N}^*$ depending only on Ω, Λ and $\mathcal{A}$ such that the following is true. Take

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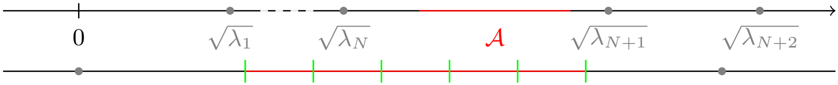

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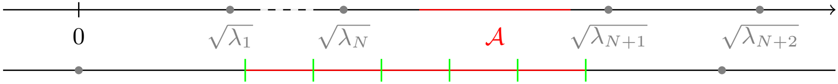

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As an example, let us consider the 1D case with $\varepsilon = 1$ and $\sigma = 0$.

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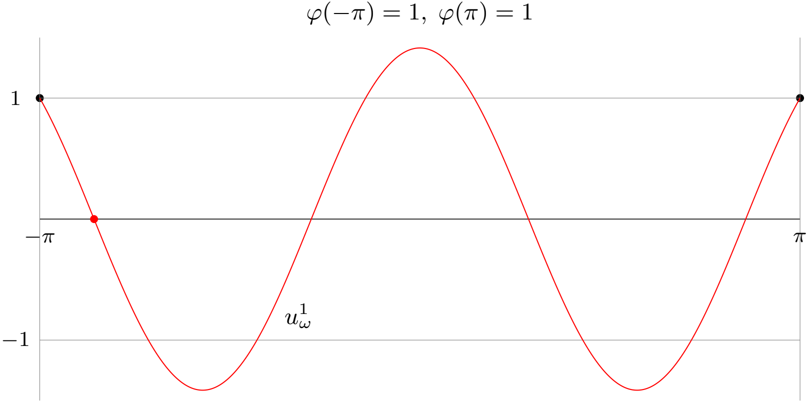

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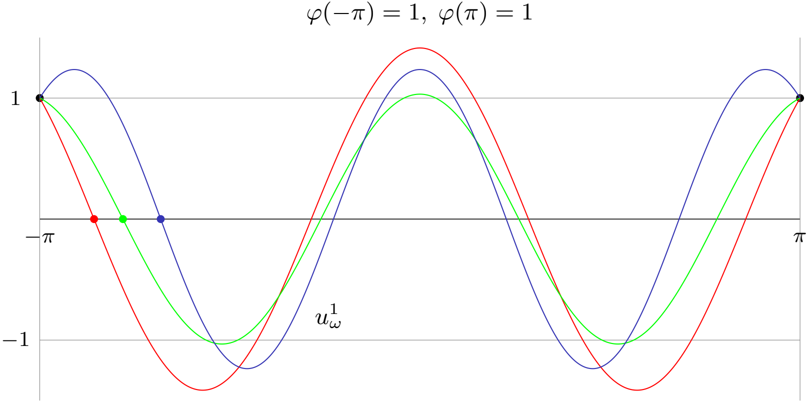

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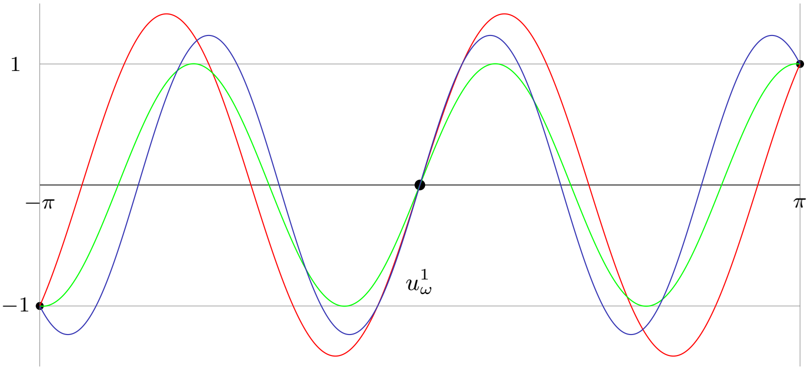

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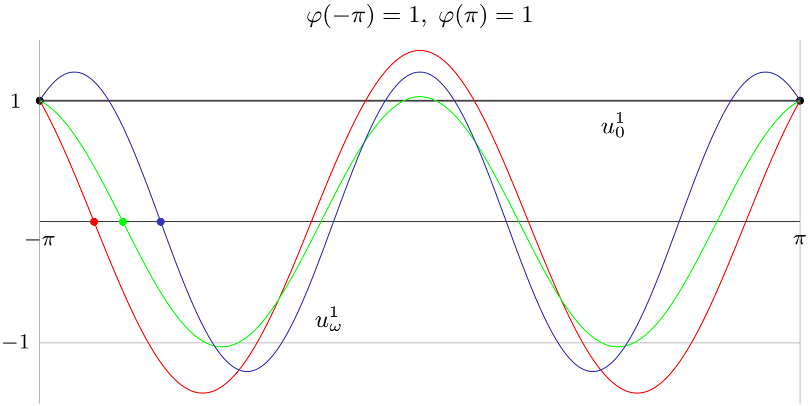

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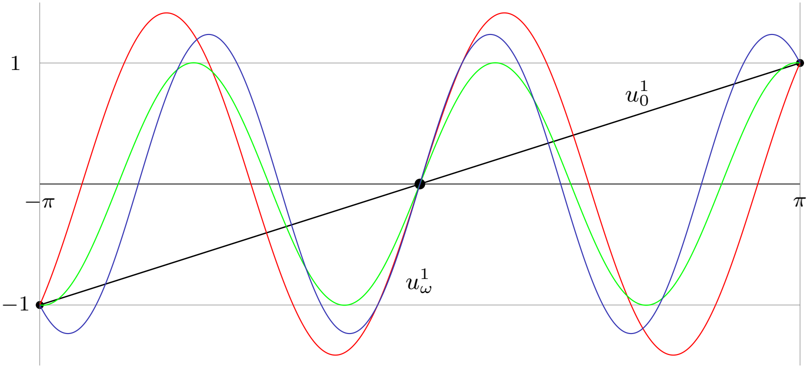

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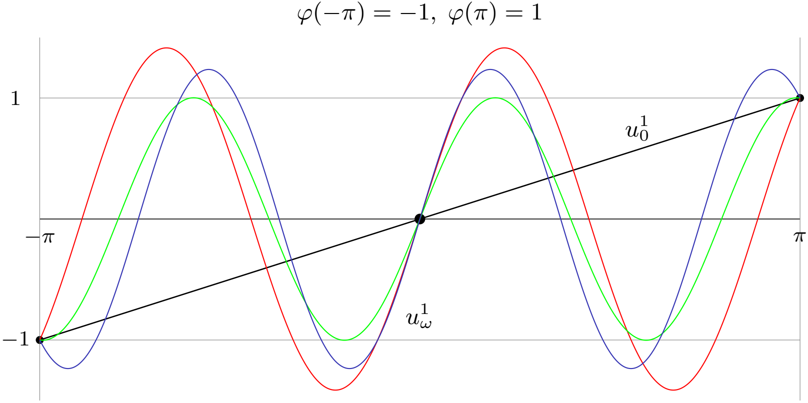

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How to pass from 0 to ω?

Lemma

The map $\mathbb{C} \setminus \sqrt{\Sigma} \longrightarrow C^1(\overline{\Omega})$, $\omega \mapsto u^i_\omega$ *is holomorphic.*

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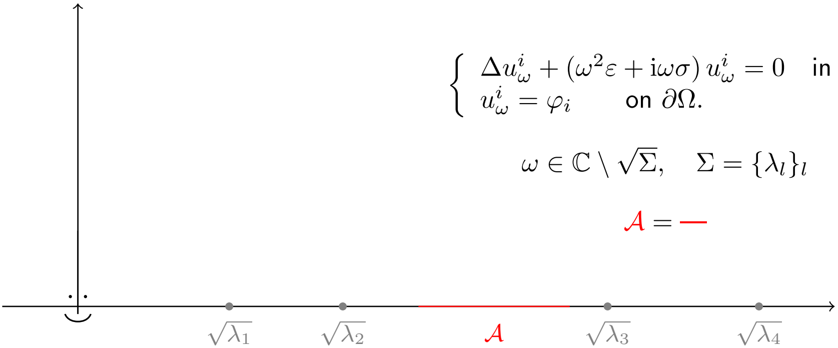

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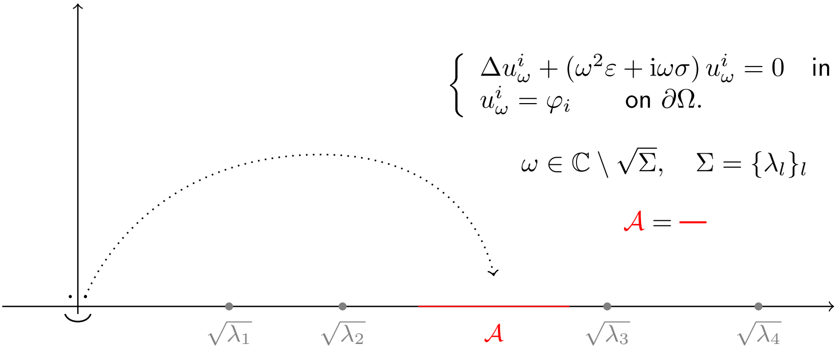

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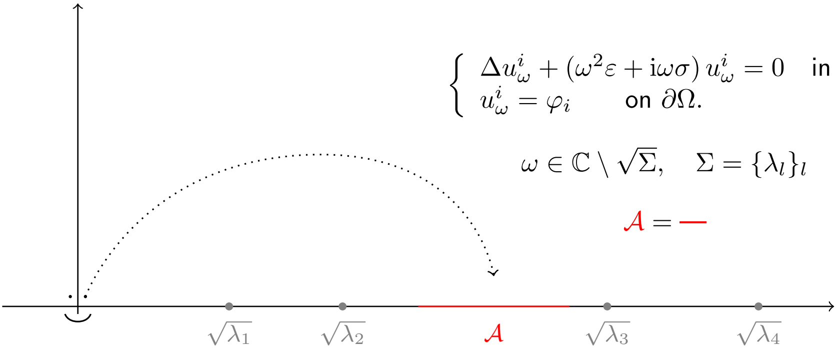

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$$\omega \in \mathbb{C} \setminus \sqrt{\Sigma}, \quad \Sigma = \{\lambda_l\}_l$$

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The map $\mathbb{C} \setminus \sqrt{\Sigma} \longrightarrow C^1(\overline{\Omega})$, $\omega \mapsto u_\omega^i$ is holomorphic.

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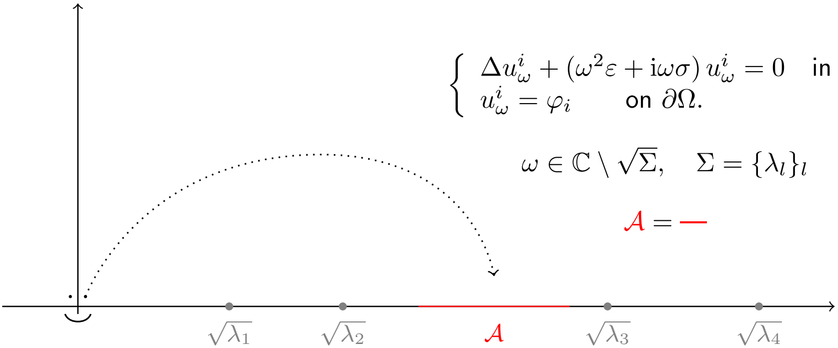

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How many frenquencies are needed?

Theorem (GSA and Capdeboscq, CM 2016)

Take $\varphi = 1$. Assume that σ and ε are real analytic. The set

$$\left\{(\omega_1, \dots, \omega_{d+1}) \in \mathcal{A}^{d+1} : \min_{\overline{\Omega}}\left(\left|u^{\varphi}_{\omega_1}\right| + \cdots + \left|u^{\varphi}_{\omega_{d+1}}\right|\right) > 0\right\}$$

is open and dense in $\mathcal{A}^{d+1}$.
In other words, (almost any) $d+1$ frequencies are ok.

Proof.

- Classical elliptic regularity theory implies that u^{φ}_{ω} is real analytic
- The set $X = \{x \in \Omega : \left|u^{\varphi}_{\omega_1}\right| = \cdots = \left|u^{\varphi}_{\omega_l}\right| = 0\}$ is an analytic variety
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Some related works

- Ammari et al. (2016) have successfully adapted this method to

$$\mathrm{div}((\omega\varepsilon + \mathrm{i}\sigma)\nabla u_\omega^i) = 0.$$

- In 2D, everything works with $a \in C^{0,\alpha}(\Omega;\mathbb{R}^{2\times 2})$ and

$$\mathrm{div}(a\,\nabla u_\omega^i) + (\omega^2\varepsilon + \mathrm{i}\omega\sigma)u_\omega^i = 0$$

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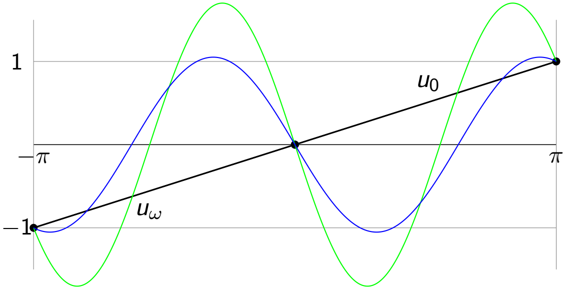

Theorem (GSA, ARMA 2016)

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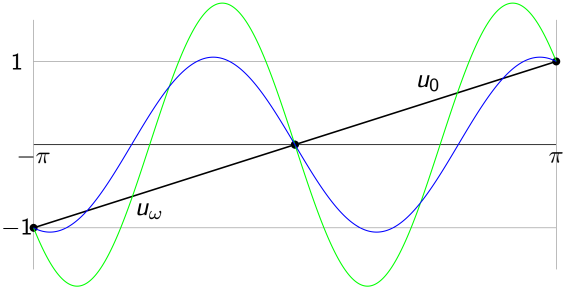

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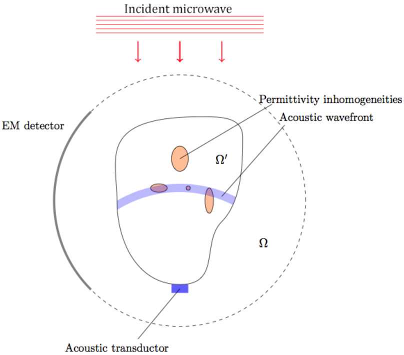
Acousto-electromagnetic tomography (Ammari et al., 2012)



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- Internal data:

$$\psi_\omega = |u_\omega|^2 \nabla \varepsilon$$

- Linearised problem:

$$D\psi_\omega[\varepsilon](\rho) \mapsto \rho$$

In order to have well-posedness of the linearised inverse problem we need

$$\|D\psi_\omega[\varepsilon](\rho)\| \geq C \|\rho\|, \qquad \rho \in H^1(\Omega),$$

or equivalently $\ker D\psi_\omega[\varepsilon] = \{0\}$.

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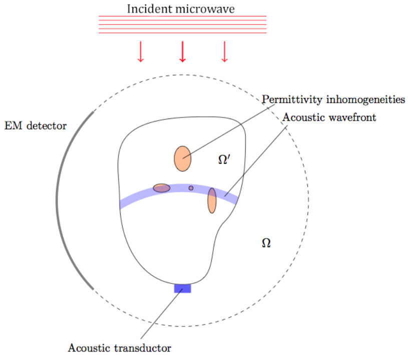

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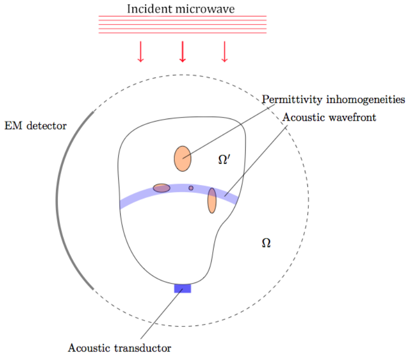

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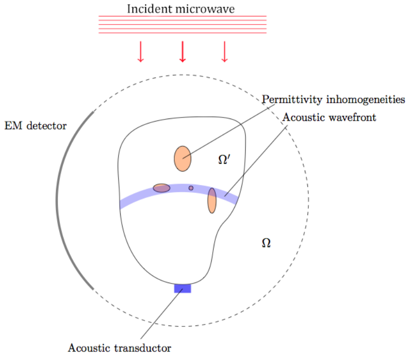

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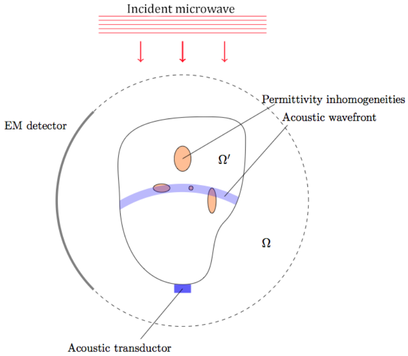

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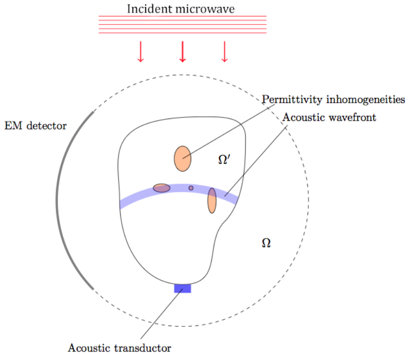

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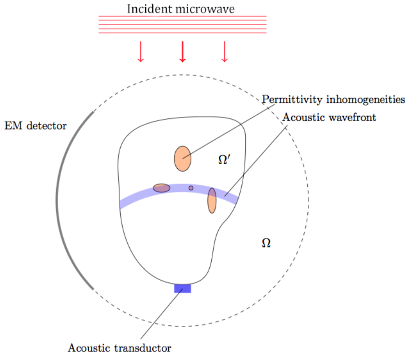

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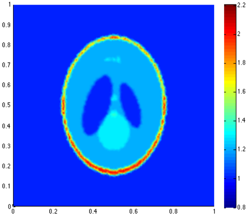

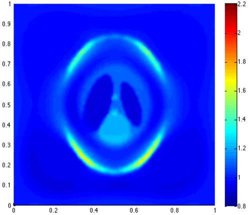

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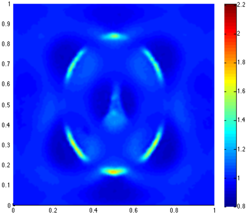

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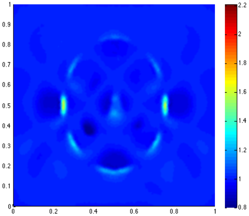

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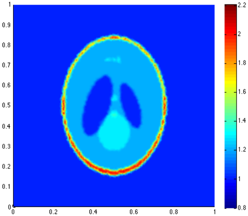

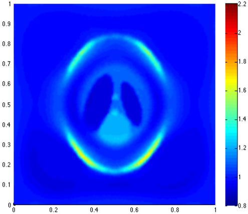

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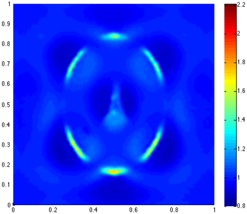

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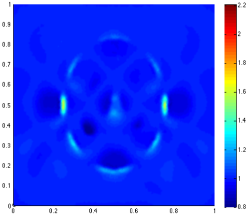

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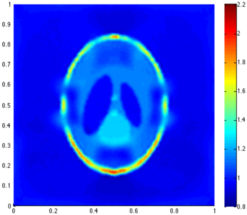

(d) $K = \{10, 15, 20\}$

Outline of the talk

1. The conductivity equation
2. The Helmholtz equation
3. The Maxwell's equations

Constraints for Maxwell's equations

- MREIT [Seo et al., 2012, Bal and Guo, 2013]

$$\begin{cases} \mathrm{curl} E^i = \mathrm{i}\omega H^i & \text{in } \Omega, \\ \mathrm{curl} H^i = -\mathrm{i}(\omega\varepsilon + \mathrm{i}\sigma)E^i & \text{in } \Omega, \\ E^i \times \nu = \varphi_i \times \nu & \text{on } \partial\Omega. \end{cases}$$

$$H^i(x) \quad \xrightarrow{?} \quad \varepsilon, \sigma$$

- The relevant constraint in this case is

$$|\det\left[E^1(x) \quad E^2(x) \quad E^3(x)\right]| > 0.$$

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- The multi-frequency method discussed above works as well [GSA, JDE 2015].
- In both cases, the regularity of the solutions is a fundamental ingredient.

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Regularity for Maxwell's equations

Theorem (GSA, 2016)

Assume that

$$\varepsilon, \mu \in C^{0,\alpha}(\overline{\Omega}; \mathbb{C}^{3\times 3}), \qquad \|(\mu, \varepsilon)\|_{C^{0,\alpha}} \leq \Lambda$$

and that are uniformly elliptic (constant Λ*). Take* $J_e, J_m \in C^{0,\alpha}(\overline{\Omega}; \mathbb{C}^3)$ *and* $G \in C^{1,\alpha}(\mathrm{curl}, \Omega)$*. Let* $(E, H) \in H(\mathrm{curl}, \Omega)^2$ *be a weak solution of*

$$\begin{cases} \mathrm{curl} H = \mathrm{i}\omega\varepsilon E + J_e & \text{in } \Omega, \\ \mathrm{curl} E = -\mathrm{i}\omega\mu H + J_m & \text{in } \Omega, \\ E \times \nu = G \times \nu & \text{on } \partial\Omega, \end{cases}$$

Then $E, H \in C^{0,\alpha}(\overline{\Omega}; \mathbb{C}^3)$ *and*

$$\|(E, H)\|_{C^{0,\alpha}} \leq C\big(\|(E, H)\|_{L^2(\Omega;\mathbb{C}^3)^2} + \|G\|_{C^{1,\alpha}(\mathrm{curl},\Omega)} + \|(J_e, J_m)\|_{C^{0,\alpha}(\overline{\Omega};\mathbb{C}^3)^2}\big)$$

for some constant C *depending only on* Ω, Λ *and* ω.

Take-home message: regularity for Maxwell is exactly as in the elliptic case!

Regularity for Maxwell's equations

Theorem (GSA, 2016)

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and that are uniformly elliptic (constant Λ). Take $J_e, J_m \in C^{0,\alpha}(\overline{\Omega}; \mathbb{C}^3)$ and $G \in C^{1,\alpha}(\mathrm{curl}, \Omega)$. Let $(E, H) \in H(\mathrm{curl}, \Omega)^2$ be a weak solution of

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Then $E, H \in C^{0,\alpha}(\overline{\Omega}; \mathbb{C}^3)$ and

$$\|(E, H)\|_{C^{0,\alpha}} \le C\big(\|(E, H)\|_{L^2(\Omega;\mathbb{C}^3)^2} + \|G\|_{C^{1,\alpha}(\mathrm{curl},\Omega)} + \|(J_e, J_m)\|_{C^{0,\alpha}(\overline{\Omega};\mathbb{C}^3)^2}\big)$$

for some constant C depending only on Ω, Λ and ω.

Take-home message: regularity for Maxwell is exactly as in the elliptic case!

Conclusions

- The inversion in quantitative hybrid imaging often requires the solutions to the direct problem to satisfy certain non-zero constraints.
- It is in general difficult to enforce these constraints a priori (independently of the unknown coefficients), but certain techniques are available:
  - The Radó-Kneser-Choquet theorem and its generalizations (only in 2D, counterexamples in 3D)
  - CGO solutions
  - The Runge approximation
  - The multi-frequency approach
- Are these constraints *generically* satisfied for a fixed number of boundary conditions?

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