Calderón's inverse problem with a finite number of measurements

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IAS, HKUST, March 12-16, 2018

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Summary

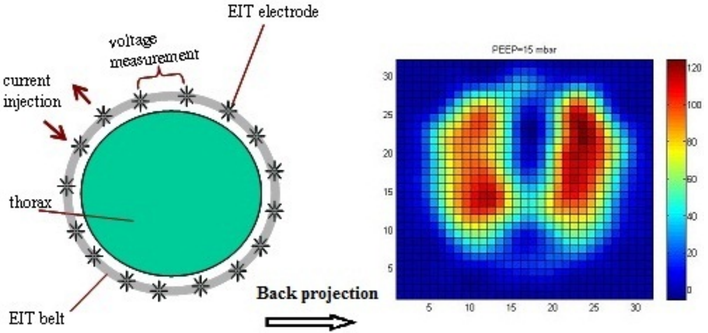
- Intro – Motivations
- Nonlinear problem: global uniqueness, Lipschitz stability and reconstruction
- Linearized problem: compressed sensing

G. S. Alberti, M. Santacesaria
Infinite dimensional compressed sensing from anisotropic measurements and applications to inverse problems in PDE, preprint arXiv:1710.11093.

G. S. Alberti, M. Santacesaria
Calderón's inverse problem with a finite number of measurements, preprint arXiv:1803.04224.

Electrical Impedance Tomography (EIT)

Monitoring lung ventilation distribution



credits: Zhao et al. Crit Care. 2010

Physical modeling – Calderón's problem

- $D \subset \mathbb{R}^d$, $d \geq 2$: bounded Lipschitz domain
- $\sigma \in L^\infty(D)$, $\lambda^{-1} \leq \sigma \leq \lambda$: unknown conductivity
- Conductivity equation:

$$\begin{cases} -\operatorname{div}(\sigma \nabla u) = 0 & \text{in } D, \\ u = f & \text{on } \partial D. \end{cases}$$

- Dirichlet-to-Neumann (DN) map $\Lambda_\sigma : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$:

$$f \longmapsto \sigma \frac{\partial u}{\partial \nu}|_{\partial D}$$

Calderón's problem

Given Λ_σ, determine σ in D.

Some known results

Basic questions:

- Uniqueness: injectivity of $\sigma \mapsto \Lambda_\sigma$
- stability estimates: continuity of $\Lambda_\sigma \mapsto \sigma$
- reconstruction algorithm

Fundamental contributions by: Calderón, Sylvester–Uhlmann, Novikov, Nachman, Alessandrini, Astala–Päivärinta, Haberman–Tataru and many others.

Usual reduction to the Gel'fand-Calderón inverse problem for the Schrödinger equation

$$(-\Delta + q)u = 0 \quad \text{in } D, \qquad \Lambda_q(u|_{\partial D}) = \left.\frac{\partial u}{\partial \nu}\right|_{\partial D},$$

which will be considered for the rest of the talk.

A finite number of measurements

$$\begin{cases} (-\Delta + q)u = 0 & \text{in } D, \\ u = f & \text{on } \partial D, \end{cases} \qquad \Lambda_q(f) = \frac{\partial u}{\partial \nu}\Big|_{\partial D}.$$

- Most results need an infinite number of measurement.
- The only exception is the reconstruction of a polygon from one measurement [Friedman-Isakov 1989] (see also [Blasten-Liu 2017]).

"Realistic" Calderón's problem

$$\{(f_l, \Lambda_q(f_l))\}_{l=1,\dots,N} \qquad \rightsquigarrow \qquad q$$

A priori assumptions: $q \in \mathcal{W}_R$ if

- $q \in \mathcal{W}$: known finite dimensional subspace of $L^\infty(D)$;
- 0 is not a Dirichlet eigenvalue for $-\Delta + q$ in D;
- $\|q\|_{L^\infty(D)} \leq R$ for some $R > 0$.

Nonlinear prolem - global uniqueness

Theorem 1 (GSA, M. Santacesaria (2018))

Take $d \geq 3$ and let $D \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain and $\mathcal{W} \subseteq L^\infty(D)$ be a finite dimensional subspace. There exists $N \in \mathbb{N}$ such that for any $R > 0$ and $q_1 \in \mathcal{W}_R$, the following is true.

There exist $\{f_l\}_{l=1}^N \subseteq H^{1/2}(\partial D)$ such that for any $q_2 \in \mathcal{W}_R$, if

$$\Lambda_{q_1} f_l = \Lambda_{q_2} f_l, \qquad l = 1, \dots, N,$$

then

$$q_1 = q_2.$$

Sketch of the proof 1

- WLOG: $D \subseteq [0,1]^d$ and extend functions by zero
- Alessandrini's identity:

$$\langle g, (\Lambda_q - \Lambda_0) f \rangle_{H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)} = \int_D q\, u_g^0 u_f^q \, dx$$

- Use CGO $g(x) = e^{\zeta_2^k \cdot x}$ and $f(x) = e^{\zeta_1^k \cdot x}(1 + r^k(x))$ for $k \in \mathbb{Z}^d$, with

$$\zeta_j^k \cdot \zeta_j^k = 0, \qquad \zeta_1^k + \zeta_2^k = -2\pi i k, \qquad \|r^k\|_{L^2([0,1]^d)} \leq c/t_k$$

- Order the frequencies: $\rho \colon l \in \mathbb{N} \mapsto k_l \in \mathbb{Z}^d$ (bijection)
- Define the nonlinear operator $U \colon L^\infty([0,1]^d) \to \ell^\infty$ by

$$(U(q))_l = \int_D q(x) e^{-2\pi i k_l \cdot x}(1 + r^{k_l}(x))\, dx$$

- $U = F + B$, where
  - F Fourier transform
  - B is a contraction (t_k large)

Sketch of the proof 2

- Define the nonlinear operator $U\colon L^\infty(D) \to \ell^\infty$ by

$$(U(q))_l = \int_D q(x) e^{-e\pi i k_l \cdot x}(1 + r^{k_l}(x))\, dx, \qquad U = F + B$$

- Assume that $\Lambda_{q_1} f_l = \Lambda_{q_2} f_l$ for $l = 1, \ldots, N$
- Then $(P_N U)(q_1) = (P_N U)(q_2)$
- Using that B is a contraction we obtain $q_1 = q_2$, since

$$\begin{aligned} \|q_1 - q_2\|_{L^2} &= \|F(q_1 - q_2)\|_{\ell^2} \\ &\le \|P_N^\perp F(q_1 - q_2)\|_{\ell^2} + \|P_N(B(q_2) - B(q_1))\|_{\ell^2} \\ &\le \|P_N^\perp F(q_1 - q_2)\|_{\ell^2} + \frac{1}{2}\|q_1 - q_2\|_{L^2}, \end{aligned}$$

provided that N is chosen so that

$$\|P_N^\perp F P_{\mathcal{W}}\|_{L^2([0,1]^d) \to \ell^2} \le \frac{1}{4}.$$

On the number of measurements N

- The number of measurements N depends only on $\mathcal{W}$ through

$$\|P_N^\perp F P_{\mathcal{W}}\|_{\mathcal{H}\to\ell^2} \leq 1/4.$$

- Relation with sampling theory: how many Fourier measurements does one need to reconstruct a function in $\mathcal{W}$?
- It allows for an explicit calculation of N:
  - bandlimited potentials

$$N = \dim \mathcal{W}$$

  - piecewise constant potentials

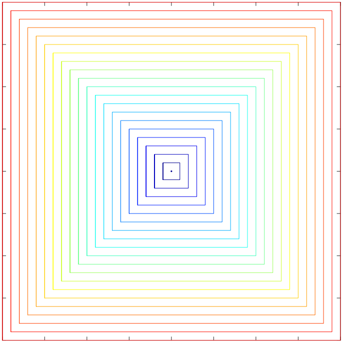
$$N = O((\dim \mathcal{W})^4)$$

    (up to log factors, and possibly not optimal)
  - low-scale wavelets

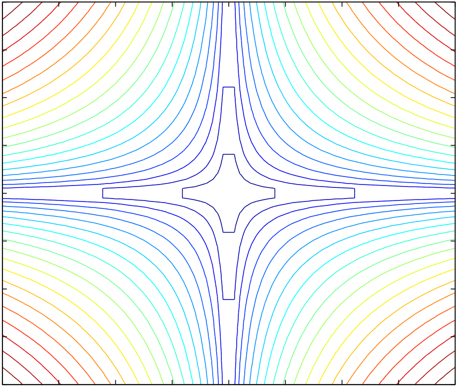
$$N = O(\dim \mathcal{W})$$

    (up to log factors, proven only in 1D, but easy generalization)
- The ordering of $\mathbb{Z}^d$ is crucial

Possible orderings of $\mathbb{Z}^d$



(a) Linear ordering



(b) Hyperbolic ordering (Jones, Adcock, Hansen, 2017)

Lipschitz stability

Theorem 2 (GSA, M. Santacesaria (2018))

Take $d \geq 3$ and let $D \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain and $\mathcal{W} \subseteq L^\infty(D)$ be a finite dimensional subspace. There exists $N \in \mathbb{N}$ such that for every $R, \alpha > 0$ and $q_1 \in \mathcal{W}_R$, the following is true.

There exist $\{f_l\}_{l=1}^N \subseteq H^{1/2}(\partial D)$ such that for every $q_2 \in \mathcal{W}_R$, we have

$$\|q_2 - q_1\|_{L^2(D)} \leq e^{CN^{\frac{1}{2}+\alpha}} \left\| (\Lambda_{q_2} f_l - \Lambda_{q_1} f_l)_{l=1}^N \right\|_{H^{-1/2}(\partial D)^N}$$

for some $C > 0$ depending only on D, R and α.

- Several authors studied stability estimates with piece-wise constant unknowns with the full DN map (Alessandrini, Beretta, Francini, Gaburro, de Hoop, Sincich, Vessella, Zhai, ...).
- The exponential $e^{CN^{\frac{1}{2}+\alpha}}$ is consistent with previous work (Rondi, Mandache) and is related to the severe ill-posedness of this IP.

Corollary for the Calderón's problem

We say that $\sigma \in \mathcal{W}_\lambda$ if

- $\sigma \in W^{2,\infty}(\Omega)$,
- $\frac{\Delta\sqrt{\sigma}}{\sqrt{\sigma}} \in \mathcal{W}$,
- $\lambda^{-1} \leq \sigma \leq \lambda$ in Ω for some $\lambda \geq 1$,
- and $\sigma = 1$ in a neighborhood of $\partial\Omega$.

Corollary 3 (GSA, M. Santacesaria (2018))

Take $d \geq 3$ and let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain and $\mathcal{W} \subseteq L^\infty(\Omega)$ be a finite dimensional subspace. There exists $N \in \mathbb{N}$ such that for any $\lambda > 1$, $\alpha > 0$ and $\sigma_1 \in \mathcal{W}_\lambda$, the following is true. There exist $\{f_l\}_{l=1}^N \subseteq H^{1/2}(\partial\Omega)$ such that for any $\sigma_2 \in W_\lambda$, we have

$$\|\sigma_2 - \sigma_1\|_{L^2(\Omega)} \leq e^{CN^{\frac{1}{2}+\alpha}} \left\| (\Lambda_{\sigma_2} f_l - \Lambda_{\sigma_1} f_l)_{l=1}^N \right\|_{H^{-1/2}(\partial\Omega)^N}$$

for some $C > 0$ depending only on Ω, λ and α.

Nonlinear reconstruction algorithm

- Set $\mathcal{W}_R := \{q \in \mathcal{W} : \|q\|_{L^\infty([0,1]^d)} \leq R\}$ (definition changed!)
- Projection $P_{\mathcal{W}_R} \colon L^2([0,1]^d) \to \mathcal{W}_R$
- Define the nonlinear operator $A : \mathcal{W}_R \to \mathcal{W}_R$ by

$$A(q') = P_{\mathcal{W}_R}\big(F^{-1}y - F^{-1}P_N B(q') + F^{-1}P_N^{\perp} F q'\big),$$

  where $y = P_N U(q)$ is the measurement.
- q is a fixed point of A, namely:

$$\begin{aligned} A(q) &= P_{\mathcal{W}_R}\big(F^{-1}P_N U(q) - F^{-1}P_N B(q) + F^{-1}P_N^{\perp} F q\big) \\ &= P_{\mathcal{W}_R}\big(F^{-1}P_N F(q) + F^{-1}P_N^{\perp} F q\big) \\ &= q \end{aligned}$$

- A is a contraction:

$$\|A(q_2) - A(q_1)\|_{L^2([0,1]^d)} \leq \frac{3}{4}\|q_2 - q_1\|_{L^2([0,1]^d)}, \quad q_1, q_2 \in \mathcal{W}_R.$$

Nonlinear reconstruction algorithm

- Define the nonlinear operator $A : \mathcal{W}_R \to \mathcal{W}_R$ by

$$A(q') = P_{\mathcal{W}_R}\big(F^{-1}y - F^{-1}P_N B(q') + F^{-1}P_N^{\perp} F q'\big),$$

where $y = P_N U(q)$ is the measurement.
- $A(q) = q$ and A is a contraction.

Choose any $q_0 \in \mathcal{W}_R$ and set $q_n = A(q_{n-1})$. By the Banach fixed point theorem:

$$\|q - q_n\|_{L^2([0,1]^d)} \leq 4\left(\frac{3}{4}\right)^n \|q_1 - q_0\|_{L^2([0,1]^d)}$$

Comments:

- q_0 is any initial guess
- guaranteed global exponential convergence to q
- only a finite number of measurements are required

Open questions

- Two-dimensional case
- Nonlinear finite dimensional manifolds $\mathcal{W}$
- Is it possible to choose $\{f_l\}_l$ independently of q?
- Boundary determination of σ
- Discrete models (CEM), numerical implementation
- Extensions to other infinite dimensional IP
- Compressed sensing
  - Number of measurements proportional to sparsity (i.e. number of nonzero components) of q
  - Measurements are taken at random
  - Solution obtained by ℓ^1 minimization
  - Successful recovery with high probability
  - So far: linearized setting
  - Nonlinear problem?

Linearized inverse problem

- Assume $q = q_0 + \delta q$:
  - $q_0 \in H^s([0,1]^d)$ known ($s > \frac{d}{2}$)
  - $\delta q \in L^2([0,1]^d)$ *small* and sparse
- Alessandrini's identity:

$$\langle f_1, (\Lambda_q - \Lambda_{q_0}) f_2 \rangle = \int_D \delta q \, u_1 u_2^0 \, dx \approx \int_D \delta q \, u_1^0 u_2^0 \, dx$$

  where $(-\Delta + q_0) u_l^0 = 0$ in D, $u_l^0|_{\partial D} = f_l$
- Choose CGO $u_i^0(x) = e^{\zeta_i^{k_l} \cdot x}(1 + r_i^{k_l}(x))$ so that

$$u_1^0(x) u_2^0(x) = e^{-2\pi i k_l \cdot x}(1 + r_1^{k_l}(x))(1 + r_2^{k_l}(x))$$

- Define the linear forward map $U : L^2([0,1]^d) \to \ell^2$ by

$$(U \, \delta q)_l = \int_D \delta q \, e^{-2\pi i k_l \cdot x}(1 + r_1^{k_l}(x))(1 + r_2^{k_l}(x)) \, dx$$

Theorem 4 (GSA, M. Santacesaria (2017))

- $\{\varphi_j\}_{j\in\mathbb{N}}$: ONB of $L^2([0,1]^d)$ of (suitable) wavelets
- $\delta q = \sum_j (\delta q)_j \varphi_j$ sparse: $\operatorname{supp}(\delta q)_j \subseteq \{1,\dots,M\}$, $\#\operatorname{supp}(\delta q)_j = s$
- $N \in \mathbb{N}$ is chosen big enough depending on M and s (as before)
- Sample m indices $l_1,\dots,l_m$ indipendently from $\{1,\dots,N\}$ according to the probability distribution $\nu_l \propto \lceil N/l \rceil$.
- Take $\omega \geq 1$: the number of measurements m satisfies

$$m \geq C\,\omega^2\, s \log^2 N.$$

Let $g \in L^2([0,1]^d)$ be a minimizer of

$$\inf_{g\in L^2([0,1]^d)} \|(g_j)_j\|_1 \quad \text{subject to } (Ug)_{l_i} = (U\,\delta q)_{l_i},\ i = 1,\dots,m.$$

Then, with probability exceeding $1 - e^{-\omega}$, we have $g = \delta q$.

Extensions to noisy measurements and *compressible* unknown

Comments on the proof

- The standard theory of compressed sensing (Candes, Donoho, Tao, Romberg):
  - finite dimension $\mathbb{C}^N$
  - random or isometric measurements (U unitary)
- Generalization to infinite dimension (Adcock, Hansen, Poon, Roman), with unitary operators U
- Motivated by inverse problems in PDE (no complete freedom in the measuring process), we proved a general CS result:
  - Hilbert space $\mathcal{H}$, finite and infinite dimension
  - the operator
    $$U\colon \mathcal{H} \to \ell^2, \qquad g \mapsto (g, \psi_l)_l$$
    is bounded, injective and with bounded inverse (not unitary)
  - in other words, the sparsifying system $\{\varphi_j\}_j$ and the measuring system $\{\psi_l\}_l$ are only *frames* of $\mathcal{H}$, and not necessarily ONB
- Theorem 3 is simply a corollary of this general result, which can be applied to many other infinite dimensional IP.

Conclusions

New (?) approach to some inverse problems for PDE with a finite number of measurements based on ideas from applied harmonic analysis, sampling theory and compressed sensing.

Some (other) perspectives:

- Study other linear/linearized problems.
- Full nonlinear problem with CS

Thank you!