UniGe | MaLGa

Regularity theory for Maxwell's equations

Giovanni S. Alberti

MaLGa - Machine learning Genova Center
University of Genoa

13 January 2021

Problem formulation

Time-harmonic Maxwell's equations

$$\begin{cases} \operatorname{curl} H = -i(\omega\varepsilon + i\sigma)E + J_e & \text{in } \Omega, \\ \operatorname{curl} E = i\omega\mu H + J_m & \text{in } \Omega, \\ E \times \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

with

$$E, H \in H(\operatorname{curl}, \Omega) = \{F \in L^2(\Omega; \mathbb{C}^3) : \operatorname{curl} F \in L^2(\Omega; \mathbb{C}^3)\}.$$

Main regularity questions:

- $E, H \in H^1$
- $E, H \in C^{0,\alpha}$
- $E, H \in H^k$, $\quad E, H \in C^{k,\alpha}$

References

C. Weber.
Regularity theorems for Maxwell's equations.
Math. Methods Appl. Sci., 3(4):523–536, 1981.

Jukka Saranen.
On an inequality of Friedrichs.
Math. Scand., 51(2):310–322 (1983), 1982.

M. Costabel.
A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains.
Mathematical Methods in the Applied Sciences, 12(4):365–368, 1990.

Hong-Ming Yin.
Regularity of weak solution to Maxwell's equations and applications to microwave heating.
J. Differential Equations, 200(1):137–161, 2004.

Peter Kuhn and Dirk Pauly.
Regularity results for generalized electro-magnetic problems.
Analysis (Munich), 30(3):225–252, 2010.

Giovanni S. Alberti and Yves Capdeboscq.
Elliptic regularity theory applied to time harmonic anisotropic Maxwell's equations with less than Lipschitz complex coefficients.
SIAM J. Math. Anal., 46(1):998–1016, 2014.

Manas Kar and Mourad Sini.
An $H^{s,p}(\mathrm{curl}; \Omega)$ estimate for the Maxwell system.
Math. Ann., 364(1-2):559–587, 2016.

G. S. Alberti and Y. Capdeboscq.
Lectures on elliptic methods for hybrid inverse problems.
Société Mathématique de France, Paris, 2018.

Giovanni S. Alberti.
Hölder regularity for Maxwell's equations under minimal assumptions on the coefficients.
Calc. Var. Partial Differential Equations, 57(3), 2018.

Basang Tsering-xiao and Wei Xiang.
Regularity of solutions to time-harmonic Maxwell's system with lower than Lipschitz coefficients.
Nonlinear Anal., 192:111693, 29, 2020.

Outline

Interior regularity

Global regularity

What else?

Interior regularity

We consider the regularity of the solutions

$$E, H \in H(\mathrm{curl}, \Omega)$$

to

$$\begin{aligned} \mathrm{curl}\, H &= -i \overbrace{(\omega\varepsilon + i\sigma)}^{\gamma} E + J_e && \text{in } \Omega, \\ \mathrm{curl}\, E &= i\omega\mu H + J_m && \text{in } \Omega, \end{aligned}$$

in a compact set

$$K \Subset \Omega.$$

UniGe | MaLGa

Warm up

Let's consider the limit $\omega \to 0$:

$$\begin{cases} \operatorname{curl} E = i\omega\mu H \\ \operatorname{curl} H = -i(\omega\varepsilon + i\sigma)E + J_e \end{cases} \implies \begin{cases} \operatorname{curl} E = 0 \\ \operatorname{curl} H = \sigma E + J_e \end{cases}$$

Writing $E = \nabla q_E$, this yields the conductivity equation for the electric potential q_E

$$-\operatorname{div}(\sigma \nabla q_E) = \operatorname{div} J_e$$

Elliptic regularity:

- $\sigma \in W^{1,3} \implies q_E \in H^2 \implies E \in H^1$
- $\sigma \in C^{0,\alpha} \implies q_E \in C^{1,\alpha} \implies E \in C^{0,\alpha}$
- Higher regularity
- ...

Warm up 2

Let's study H^1 regularity in homogeneous isotropic media:

$$\begin{cases} \operatorname{curl} H = -i\gamma_0 E + J_e & \text{in } \Omega, \\ \operatorname{curl} E = i\omega\mu_0 H + J_m & \text{in } \Omega. \end{cases}$$

with sources

$$J_e, J_m \in H(\operatorname{div}, \Omega) = \{F \in L^2(\Omega; \mathbb{C}^3) : \operatorname{div} F \in L^2(\Omega; \mathbb{C}^3)\}.$$

Key observation:

$$\begin{cases} \operatorname{div} E = -i\gamma_0^{-1} \operatorname{div} J_e \in L^2(\Omega) \\ \operatorname{curl} E = i\omega\mu_0 H + J_m \in L^2(\Omega) \end{cases} \stackrel{?}{\Longrightarrow} E \in H^1_{\mathrm{loc}}(\Omega)$$

Gaffney-Friedrichs Inequality (without boundary)

Theorem

We have

$$H(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega) \subseteq H^1_{\mathrm{loc}}(\Omega)$$

and

$$\|\nabla F\|_{L^2(K)} \lesssim \|\mathrm{curl}\, F\|_{L^2(\Omega)} + \|\mathrm{div}\, F\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)}.$$

Proof.

- Helmholtz decomposition: $F = \nabla q + \mathrm{curl}\, \Phi$ with $\mathrm{div}\, \Phi = 0$

- By elliptic regularity applied to

$$-\Delta \Phi = \mathrm{curl}\,\mathrm{curl}\, \Phi = \mathrm{curl}\, F$$

  we obtain $\Phi \in H^2_{\mathrm{loc}}(\Omega)$, so that

$$\mathrm{curl}\, \Phi \in H^1_{\mathrm{loc}}(\Omega).$$

- By elliptic regularity applied to

$$-\Delta q = -\mathrm{div}\, \nabla q = -\mathrm{div}\, F$$

  we obtain $q \in H^2_{\mathrm{loc}}(\Omega)$, so that

$$\nabla q \in H^1_{\mathrm{loc}}(\Omega).$$

□

Basic assumptions

$$
\begin{aligned}
\operatorname{curl} H &= -i \overbrace{(\omega\varepsilon + i\sigma)}^{\gamma} E + J_e && \text{in } \Omega, \\
\operatorname{curl} E &= i\omega\mu H + J_m && \text{in } \Omega,
\end{aligned}
$$

- frequency $\omega > 0$
- the coefficients $\varepsilon, \sigma \in L^\infty\left(\Omega; \mathbb{R}^{3\times 3}\right)$ and $\mu \in L^\infty\left(\Omega; \mathbb{C}^{3\times 3}\right)$ are elliptic:

$$\Lambda^{-1} |\eta|^2 \leq \xi \cdot \varepsilon\xi, \qquad \xi \in \mathbb{R}^3,$$

$$\Lambda^{-1} |\eta|^2 \leq \xi \cdot \left(\mu + \overline{\mu}^T\right) \xi, \qquad \xi \in \mathbb{R}^3,$$

- sources $J_e, J_m \in L^2(\Omega; \mathbb{C}^3)$

H^1 regularity

$$\begin{cases} \operatorname{curl} H = -i\gamma E + J_e & \text{in } \Omega, \\ \operatorname{curl} E = i\omega\mu H + J_m & \text{in } \Omega, \end{cases}$$

Theorem

If $\varepsilon, \sigma, \mu \in W^{1,3}$ and $J_e, J_m \in H(\operatorname{div}, \Omega)$ then $E, H \in H^1_{\mathrm{loc}}(\Omega)$.

Proof.

Assume for simplicity $\varepsilon, \mu \in W^{1,\infty}$.

- Helmholtz decomposition: $E = \nabla q_E + \operatorname{curl} \Phi_E, \quad H = \nabla q_H + \operatorname{curl} \Phi_H$

- By elliptic regularity applied to

$$-\Delta \Phi_E = i\omega\mu H + J_m$$
$$-\Delta \Phi_H = -i\gamma E + J_e$$

we obtain $\Phi_E, \Phi_H \in H^2_{\mathrm{loc}}$.

- By elliptic regularity applied to

$$-\operatorname{div}(\mu \nabla q_H) = \operatorname{div}\left(\mu \operatorname{curl} \Phi_H - i\omega^{-1} J_m\right) \in L^2$$
$$-\operatorname{div}(\gamma \nabla q_E) = \operatorname{div}(\gamma \operatorname{curl} \Phi_E + iJ_e) \in L^2$$

we obtain $q_E, q_H \in H^2_{\mathrm{loc}}(\Omega)$.

□

$C^{0,\alpha}$ regularity

Theorem

If $\varepsilon, \sigma, \mu \in C^{0,\alpha}$ and $J_e, J_m \in C^{0,\alpha}$ with $\alpha \in (0, \frac{1}{2}]$, then $E, H \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$.

Proof.

The Helmholtz decomposition $E = \nabla q_E + \operatorname{curl} \Phi_E, \quad H = \nabla q_H + \operatorname{curl} \Phi_H$ yields

$$
\begin{aligned}
-\Delta \Phi_E &= i\omega\mu H + J_m & \qquad -\operatorname{div}(\mu \nabla q_H) &= \operatorname{div}\left(\mu \operatorname{curl} \Phi_H - i\omega^{-1} J_m\right) \\
-\Delta \Phi_H &= -i\gamma E + J_e & \qquad -\operatorname{div}(\gamma \nabla q_E) &= \operatorname{div}\left(\gamma \operatorname{curl} \Phi_E + i J_e\right)
\end{aligned}
$$

- H^2 regularity: $\Phi_E, \Phi_H \in H^2 \subseteq W^{1,6}$, so that $\operatorname{curl} \Phi_E, \operatorname{curl} \Phi_H \in L^6$
- $W^{1,p}$ regularity: $\nabla q_E, \nabla q_H \in L^6$, so that $E, H \in L^6$
- $W^{2,p}$ regularity: $\Phi_E, \Phi_H \in W^{2,6}$, so that $\operatorname{curl} \Phi_E, \operatorname{curl} \Phi_H \in W^{1,6} \subseteq C^{0,\frac{1}{2}}$
- Schauder estimates: $\nabla q_E, \nabla q_H \in C^{0,\alpha}$, so that $E, H \in C^{0,\alpha}$

□

Higher regularity

Higher regularity results for elliptic equations $\Longrightarrow$ Higher regularity results for Maxwell's equations

Theorem

If $\varepsilon, \sigma, \mu \in W^{N,3}$ and $J_e, J_m \in H^N(\mathrm{div}, \Omega)$ then $E, H \in H^N_{\mathrm{loc}}(\Omega)$.

Theorem

If $\varepsilon, \sigma, \mu \in C^{N,\alpha}$ and $J_e, J_m \in C^{N,\alpha}$ with $\alpha \in (0, \frac{1}{2}]$, then $E, H \in C^{,\alpha}_{\mathrm{loc}}(\Omega)$.

UniGe | MaLGa

Outline

Interior regularity

Global regularity

What else?

Elliptic boundary regularity

Key points:

1. Helmholtz decomposition of E and H:

$$E = \nabla q_E + \operatorname{curl} \Phi_E, \quad H = \nabla q_H + \operatorname{curl} \Phi_H$$

2. Elliptic regularity applied to:

$$-\Delta \Phi_E = i\omega\mu H + J_m \qquad -\operatorname{div}(\mu \nabla q_H) = \operatorname{div}\left(\mu \operatorname{curl} \Phi_H - i\omega^{-1} J_m\right)$$
$$-\Delta \Phi_H = -i\gamma E + J_e \qquad -\operatorname{div}(\gamma \nabla q_E) = \operatorname{div}(\gamma \operatorname{curl} \Phi_E + iJ_e)$$

So:

- We can use boundary elliptic regularity!
- Need boundary conditions for the potentials Φ_E, Φ_H, q_E and q_H:

$$\Phi_E \cdot \nu = 0, \qquad \Phi_H \times \nu = 0, \qquad q_E = 0 \qquad \text{on } \partial\Omega$$

Boundary conditions for q_E

- Elliptic PDE:

$$-\operatorname{div}(\gamma \nabla q_E) = \operatorname{div}(\gamma \operatorname{curl} \Phi_E + iJ_e) \qquad \text{in } \Omega$$

- The Helmholtz decomposition gives

$$q_E = 0 \qquad \text{on } \partial\Omega$$

- Dirichlet problem!

Boundary conditions for q_H

- Elliptic PDE:

$$-\operatorname{div}\left(\mu\nabla q_H\right) = \operatorname{div}\left(\mu \operatorname{curl}\Phi_H - i\omega^{-1}J_m\right) \qquad \text{in } \Omega$$

- From

$$0 = \operatorname{div}(E\times\nu) = \operatorname{curl} E\cdot\nu = i\omega\mu H\cdot\nu + J_m\cdot\nu = i\omega\mu\nabla q_H\cdot\nu + i\omega\mu \operatorname{curl}\Phi_H\cdot\nu + J_m\cdot\nu$$

we obtain

$$-\mu\nabla q_H\cdot\nu = \left(\mu \operatorname{curl}\Phi_H - i\omega^{-1}J_m\right)\cdot\nu \qquad \text{on } \partial\Omega$$

- Neumann problem!

Boundary conditions for Φ_E and Φ_H

- 6 PDEs:

$$-\Delta\Phi_E = i\omega\mu H + J_m, \qquad -\Delta\Phi_H = -i\gamma E + J_e \qquad \text{in } \Omega.$$

- 3 Boundary conditions:

$$\Phi_E \cdot \nu = 0, \qquad \Phi_H \times \nu = 0, \qquad \text{on } \partial\Omega$$

- What to do?

The flat case

- Let's focus on Φ_H:

$$-\Delta\Phi_H = -i\gamma E + J_e \qquad \text{in } \Omega, \qquad \Phi_H \times \nu = 0, \qquad \text{on } \partial\Omega.$$

- Suppose $\Omega = \{x_3 < 0\}$, so that $\nu = e_3$. Thus:

$$\Phi_H \times \nu = 0 \implies (\Phi_H)_1 = (\Phi_H)_2 = 0$$

  and

$$\operatorname{div}\Phi_H = 0 \implies \partial_1(\Phi_H)_1 + \partial_2(\Phi_H)_2 + \partial_3(\Phi_H)_3 = 0 \implies \partial_3(\Phi_H)_3 = 0 \implies \partial_\nu(\Phi_H)_3 = 0$$

- Dirichlet and Neumann problems!

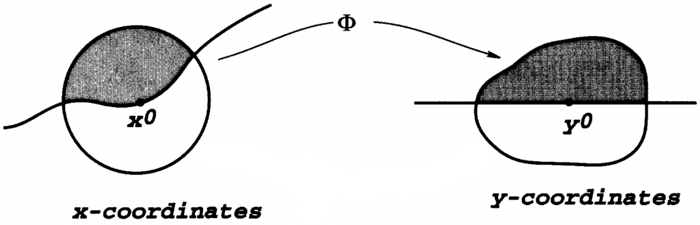
Flattening out the boundary

- Ω is locally defined by

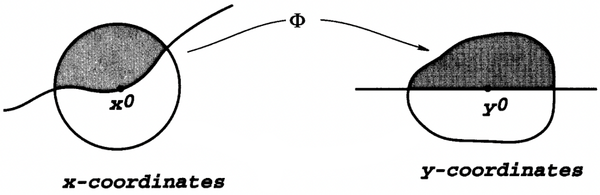
$$x_3 < \kappa(x_1, x_2)$$

- Change of coordinates $y = \Phi(x)$:

$$y_1 = x_1, \qquad y_2 = x_2, \qquad y_3 = x_3 - \kappa(x_1, x_2)$$



Piola transformation



- Setting

$$\tilde{E} = (\Phi')^{-T} E, \qquad \tilde{\gamma} = \Phi' \gamma (\Phi')^T,$$

  we have

$$\begin{cases} \operatorname{curl} E = i\omega\mu H + J_m \\ -\operatorname{div}(\gamma E) = \operatorname{div}(iJ_e) \\ E \times \nu = 0 \end{cases} \qquad \Bigg| \qquad \begin{cases} \operatorname{curl} \tilde{E} = (i\omega\mu H + J_m)^{\sim} \\ -\operatorname{div}(\tilde{\gamma} \tilde{E}) = \operatorname{div}(iJ_e) \\ \tilde{E} \times e_3 = 0 \end{cases}$$

- Same equations!

What regularity is needed?

- New PDEs

$$\operatorname{curl} \tilde{E} = (i\omega\mu H + J_m)\tilde{}, \qquad -\operatorname{div}(\tilde{\gamma}\tilde{E}) = \operatorname{div}(iJ_e), \qquad \tilde{E} \times e_3 = 0$$

  with coefficient

$$\tilde{\gamma} = \Phi'\gamma(\Phi')^T$$

- If $\partial\Omega$ is of class $C^{1,1}$, then $\Phi' \in C^{0,1}$ and
  - H^1 regularity: $\gamma \in W^{1,\infty} \implies \tilde{\gamma} \in W^{1,\infty}$
  - $C^{0,\alpha}$ regularity: $\gamma \in C^{0,\alpha} \implies \tilde{\gamma} \in C^{0,\alpha}$
- Higher regularity: $\partial\Omega$ of class $C^{N,1}$
- Non-smooth domains: many results (Buffa, Costabel, Dauge, Nicaise …)

Outline

Interior regularity

Global regularity

What else?

Other results

- Regularity only for E or H:

$$\gamma \in C^{0,\alpha} \implies E \in C^{0,\alpha}$$

- $W^{1,p}$ regularity:

$$\mu, \gamma \in W^{1,p},\ p > 3 \implies E, H \in W^{1,p}$$

- Meyers theorem:

$$\text{no additional assumptions} \implies E, H \in L^{2+\delta}$$

- Asymptotic expansions in the presence of small inhomogeneities

Maxwell regularity

=

Helmholtz decomposition + elliptic regularity