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Compressed sensing for the sparse Radon transform

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Joint work with

Alessandro Felisi
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Matteo Santacesaria
(UniGe)

S. Ivan Trapasso
(PoliTo)

Outline

The sparse Radon transform

Compressed sensing

Compressed sensing for the sparse Radon transform

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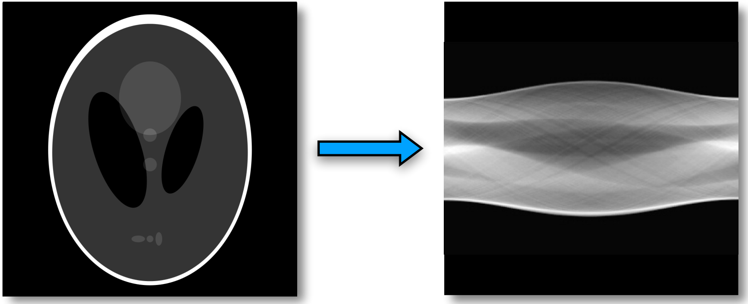
The Radon transform

$$\mathcal{R}u(\theta, s) = \int_{\theta^\perp} u(y + s\theta)dy$$



The Radon transform[1]

- Domain: $\mathcal{B}_1 = \mathrm{B}(0, 1) \subseteq \mathbb{R}^2$

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- Ill-posedness/inversion:

$$\|\mathcal{R}u\|_{L^2} \asymp \|u\|_{H^{-\frac{1}{2}}}$$

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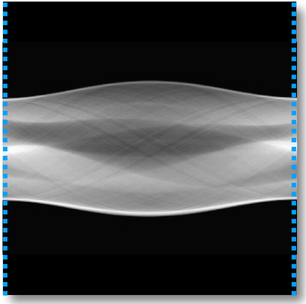

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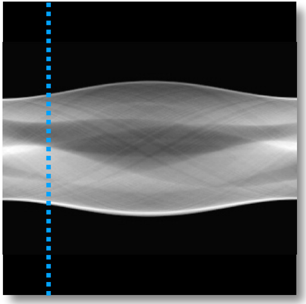

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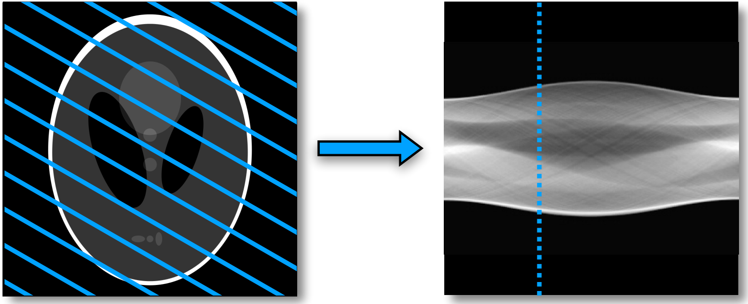

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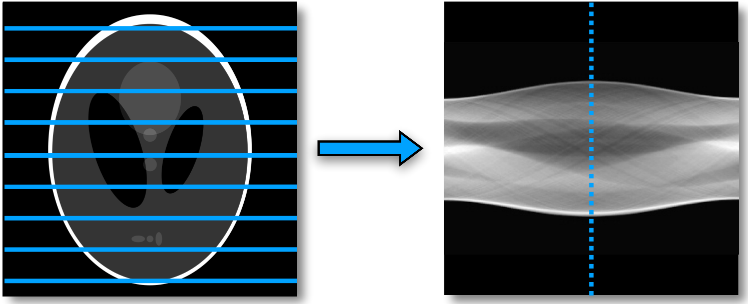
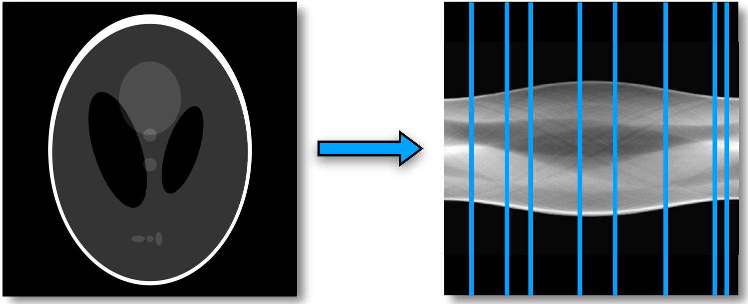


The sparse Radon transform

$$\left(\mathcal{R}u^\dagger(\theta_1,\cdot),\dots,\mathcal{R}u^\dagger(\theta_m,\cdot)\right), \quad \theta_1,\dots,\theta_m \overset{\text{i.i.d.}}{\sim} \nu \text{ uniform on } \mathbb{S}^1$$



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- Subsampled measurements $\Longrightarrow$ need a-priori information on $u^\dagger$
- Natural assumption: $u^\dagger$ is sparse

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(Some) related literature

$$\left(\mathcal{R}u^\dagger(\theta_1,\cdot),\ldots,\mathcal{R}u^\dagger(\theta_m,\cdot)\right) \in L^2(-1,1)^m \qquad \longrightarrow \qquad u^\dagger \in L^2(\mathcal{B}_1)$$

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Empirical works:

- Siltanen et al, *Statistical inversion for medical x-ray tomography with few radiographs*, 2003
- Hämäläinen et al, *Sparse Tomography*, 2013
- Jørgensen and Sidky, *How little data is enough? Phase-diagram analysis of sparsity-regularized X-ray computed tomography*, 2015
- Jørgensen, Coban, Lionheart, McDonald and Withers, *SparseBeads data: benchmarking sparsity-regularized computed tomography*, 2017
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Theoretical works:

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A theory of sparse Radon transform?

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  Compressive sensing connects the critical number of projections to the image sparsity, but does not cover CT. Empirical results suggest a similar connection.

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- From Hansen, 2017:

  We used simulations studies to provide a foundation for the use of sparsity in CT where, unlike compressed sensing, it is not possible to give rigorous proofs.

WARNING

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Main result at the end!

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SPOILER

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$$m \gtrsim \textbf{sparsity}$$

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Setup:

- Unknown signal: $u^\dagger \in \mathbb{R}^M$
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- $(Au)_l = \langle u, \psi_l \rangle$, $l = 1, \dots, m$

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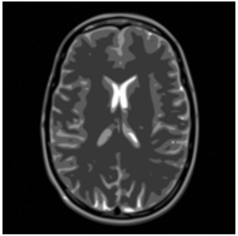

$u^\dagger$

Measured frequencies

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Problem: given $y := Au^\dagger$, retrieve the signal $u^\dagger$

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Solution: consider only **sparse** $u^\dagger$, and retrieve $u^\dagger$ in a **nonlinear** fashion

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- In practice, **compressibility**:

$$u = v + \text{small}, \qquad v \in \Sigma_s.$$

Real-world signals are compressible

Figure: Left: original image - Right: image obtained (roughly) by keeping only the 1% largest coefficients with respect to a discrete wavelet basis (JPEG-2000 compression standard)

Recovery estimate[3]

- $u^\dagger \in \mathbb{R}^M$: unknown signal
- $u^\dagger$ is s-sparse w.r.t. $\{\Phi_n\}_n$
- $(Au)_l = \langle u, \psi_l \rangle$, $l = 1, \dots, m$: subsampled isometry

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Theorem

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then, with high probability,

$$u^\dagger = u_*$$

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1. Forward map $\mathcal{R}$ affects sparsity[5]

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- Solution: many dictionaries and operators of interest are ‘compatible’

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- For $b = \frac{1}{2}$, the forward map $\mathcal{R}$ satisfies

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- Information on sparsity of $u^\dagger \Rightarrow$ information on $\mathcal{R}u^\dagger$

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2. Ill-posed problem: g-RIP

- Classical CS: Restricted Isometry Property (RIP)

$$(1-\delta)\|u\|^2 \leqslant \|Au\|_2^2 \leqslant (1+\delta)\|u\|^2, \quad u \in \Sigma_s$$

for m sufficiently large.

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Then, with high probability,

$$u_* = u^\dagger$$

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A few comments

- This theorem is a particular case of an abstract result dealing with:
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  - Hilbert space-valued measurements
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- Explicit estimates with
  - noisy data
  - compressible (and not sparse) $u^\dagger$
  - regularization with sampling: $m = m(\text{noise})$

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- Rigorous theory of compressed sensing for subsampled isometries (e.g. MRI)
- Empirical evidence for compressed sensing Radon transform

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Future

- Fan-beam geometry
- Wavelets $\rightarrow$ shearlets, curvelets, etc.
- Generalisation to other ill-posed problems
- Nonlinear problems
- Compressed sensing with generative models