Continuous generative neural networks

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CGNNs in one sentence

Continuous Generative Neural Networks (CGNNs) are a machine learning architecture that represent elements in infinite-dimensional function spaces and provide Lipschitz stability for inverse problems.

CGNNs in one formula

A CGNN $G\colon \mathbb{R}^{40} \to L^2((0,1)^2)$ generating a 40-dim manifold of handwritten digits.

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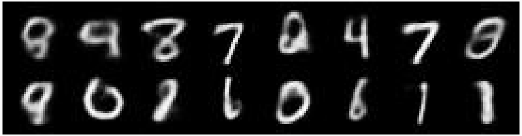
$$z \sim (\mathcal{N}(0,1))^{40} \xmapsto{\text{16 times}}$$



Outline

Generative models, inverse problems and stability

Continuous Generative Neural Networks

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X, Y Hilbert spaces, $\mathcal{F}\colon X \to Y$ possibly nonlinear

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Given $y = \mathcal{F}(x^\dagger) + \varepsilon$, determine $x^\dagger$ by solving

$$\arg\min_{x \in X} \{\|\mathcal{F}(x) - y\|^2 + R(x)\}.$$

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$$\arg\min_{x \in X} \{\|\mathcal{F}(x) - y\|^2 + R(x)\}.$$

With a generative model

$$\text{Determine } x^\dagger = G(z^\dagger) \text{ by solving}$$

$$\arg\min_{z \in Z} \{\|\mathcal{F}(G(z)) - y\|^2\},$$

where $G\colon Z \to X$ is a generator.

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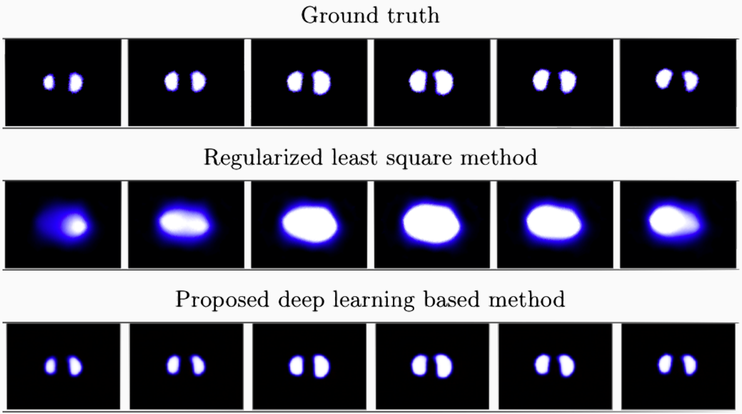
Example with electrical impedance tomography[1]



Classical approach:

$$y = F(x)$$

With Generative models:

$$y = F(G(z))$$

[1] Seo-Kim-Jargal-Lee-Harrach, A learning-based method for solving ill-posed nonlinear inverse problems: a simulation study of Lung EIT, 2019

A general Lipschitz stability result[2]

Theorem (Alberti-Arroyo-S. 2022)

If

- $M \subseteq X$ **finite-dimensional manifold**,
- and $\mathcal{F}|_M$ and $\mathcal{F}'(x)|_{T_x M}$, for $x \in M$, are injective,

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then

$$\|x_1 - x_2\|_X \leqslant C\|\mathcal{F}(x_1) - \mathcal{F}(x_2)\|_Y, \qquad x_1, x_2 \in M.$$

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Pros: higher stability, more accurate modeling, less computations.

Cons: missing theory!

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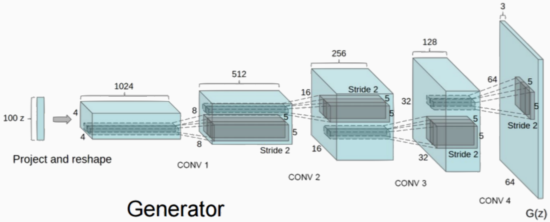

Fully Connected layer

$$y = \sigma(Fx + b)$$

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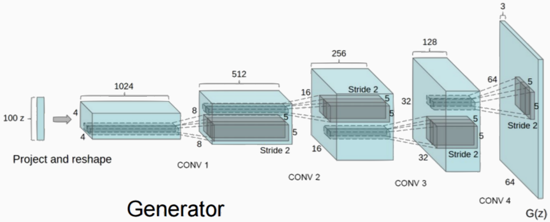

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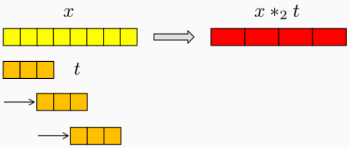
Convolutional layer (filters t_i^k)

$$(f_{out})_k = \sigma\left(\sum_{i=1}^{c} (f_{in})_i *_s t_i^k + b^k\right)$$

Strided continuous convolution

Discrete case



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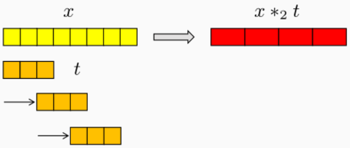

Continuous case

$$V_j \xrightarrow[\text{conv.}]{\Psi} L^2([0,1]) \xrightarrow[\text{proj.}]{P_{V_{j-1}}} V_{j-1},$$

- Ψ is a continuous convolution
- $\cdots \subseteq V_{j-1} \subseteq V_j \subseteq V_{j+1} \subseteq \cdots$: scale spaces of a wavelet multiresolution analysis

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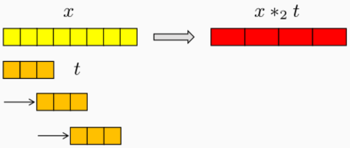

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Easy implementation. Discrete convolution (almost) for the wavelet coefficients

Discrete and Continuous Generator structure in 1D [3]

Discrete Generator

$$G : \mathbb{R}^\eta \xrightarrow[\text{f.c.}]{F\,\cdot\,+b} (\mathbb{R}^{\alpha_1})^{c_1} \xrightarrow[\text{non lin.}]{\sigma} (\mathbb{R}^{\alpha_1})^{c_1}$$

$$\xrightarrow[\text{conv.}]{\Psi_2} (\mathbb{R}^{\alpha_2})^{c_2} \xrightarrow[\text{non lin.}]{\sigma} (\mathbb{R}^{\alpha_2})^{c_2} \dots$$

$$\dots \xrightarrow[\text{conv.}]{\Psi_L} \mathbb{R}^{\alpha_L} \xrightarrow[\text{non lin.}]{\sigma} \mathbb{R}^{\alpha_L}$$

$$c_1 > c_2 > \dots > c_L$$

$$\alpha_1 < \alpha_2 < \dots < \alpha_L$$

[3] J. Bruna, S. Mallat, Invariant Scattering Convolution Networks, 2012
N. Kovachki et al., Neural Operator: Learning Maps Between Function Spaces, 2021
A. Habring, M. Holler, A generative variational model for inverse problems in imaging, 2022
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Continuous Generator structure in 1D: Formalization

$$
\begin{aligned}
G : \mathbb{R}^{\eta} &\xrightarrow[\text{f.c.}]{F\ \cdot\ +b} (V_{j_1})^{c_1} \xrightarrow[\text{non lin.}]{\sigma} (L^2([0,1]))^{c_1} \xrightarrow[\text{proj.}]{P_{(V_{j_1})^{c_1}}} (V_{j_1})^{c_1} \\
&\xrightarrow[\text{conv.}]{\Psi_2} (L^2([0,1]))^{c_2} \xrightarrow[\text{proj.}]{P_{(V_{j_2})^{c_2}}} (V_{j_2})^{c_2} \xrightarrow[\text{non lin.}]{\sigma} (L^2([0,1]))^{c_2} \\
&\xrightarrow[\text{proj.}]{P_{(V_{j_2})^{c_2}}} (V_{j_2})^{c_2} \quad \dots \quad \xrightarrow[\text{conv.}]{\Psi_L} L^2([0,1]) \xrightarrow[\text{proj.}]{P_{V_{j_L}}} V_{j_L} \\
&\xrightarrow[\text{non lin.}]{\sigma} L^2([0,1]) \xrightarrow[\text{proj.}]{P_{V_{j_L}}} V_{j_L}
\end{aligned}
$$

Main features

Discretization invariance

[4] Puthawala, Kothari, Lassas, Dokmanić, de Hoop, Globally injective ReLU networks, 2020
Hagemann, Neumayer, Stabilizing invertible neural networks using mixture models, 2021

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Injectivity[4]

Theorem (G.S.Alberti, M. S., S. Sciutto, 2022)

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Lipschitz stability for inverse problems

Theorem (G.S.Alberti, M. S., S. Sciutto, 2022)

Assume G as above, set $\mathrm{Im}(G) = M$. Then

$$\|x - y\|_X \leqslant C\|\mathcal{F}(x) - \mathcal{F}(y)\|_Y, \qquad x, y \in M.$$

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Conclusions

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- Training protocols (we used variational autoencoders)
- Architecture (convolutional; we only used the approximation coefficients)
- Manifold learning (only one chart)

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Future work

- Multiple generators for nontrivial topologies (with J. Hertrich)
- Different architectures
- How does the training affect the reconstruction performances?

[Continuous Generative Neural Networks, preprint arXiv:2205.14627]

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