Localization of point-like scatterers via sparse optimization on measures

Giovanni S. Alberti, Romain Petit, Matteo Santacesaria

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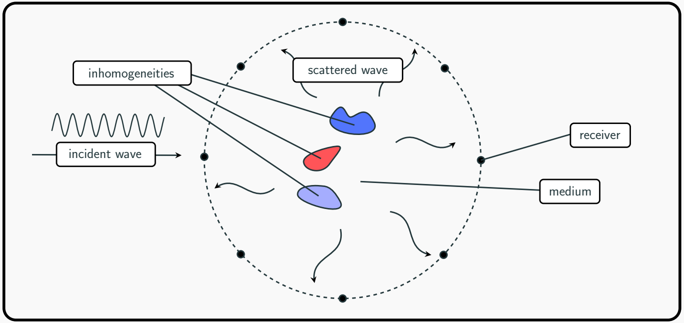
The (acoustic) inverse medium problem

Inverse problem

Reconstruct inhomogeneities from measurements of scattered waves at infinity



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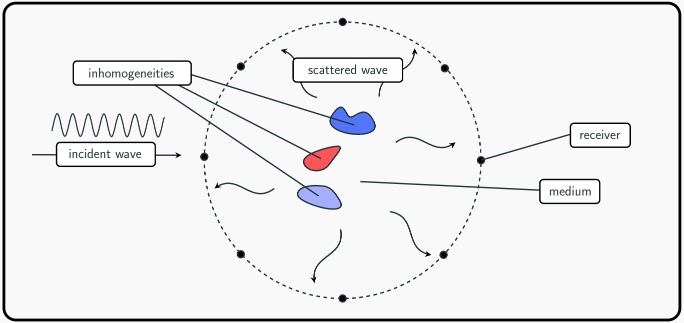


Modeling [Colton and Kress, 2012]

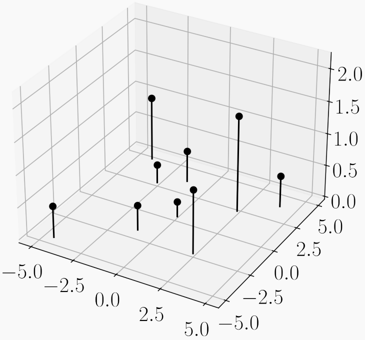
- Incident field u^{in} satisfies $\Delta u^{in} + \kappa^2 u^{in} = 0$, i.e. $u^{in}(x) = e^{i\kappa\theta\cdot x}$, $\theta \in \mathbb{S}^{d-1}$
- Total field $u = u^{in} + u^s$ satisfies $\Delta u + \kappa^2(1+\mu)u = 0$

The (acoustic) inverse medium problem

Our case

Point-like inhomogeneities:

$$\mu = \sum_{i=1}^{s} a_i \delta_{x_i}$$



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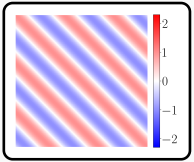
Point-like inhomogeneities [de Vries et al., 1998, Albeverio et al., 2012]



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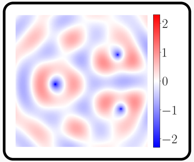
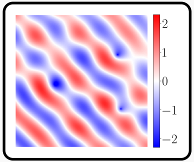


Scattered field



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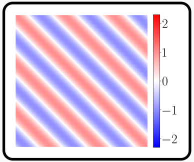

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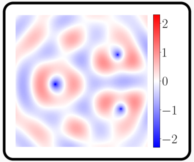

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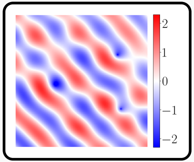

Total field

Inverse problem

- Unknown: perturbation $\mu = \sum_i a_i \delta_{x_i}$
- Observations: far field measurements $(u^\infty(\hat{x}_j, \theta_j))_{j=1}^m$ + noise

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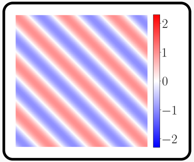

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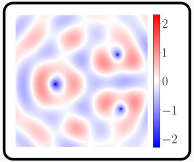

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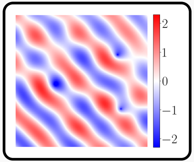

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Far field pattern [Foldy, 1945]

- Far field pattern: $u^\infty(\hat{x}, \theta) \propto \sum_i a_i u_i e^{-i\kappa \hat{x} \cdot x_i}$, $(\hat{x}, \theta) \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$
- Foldy-Lax system: $u_i = e^{i\kappa\theta \cdot x_i} + \kappa^2 \sum_{j \neq i} \Phi(x_i, x_j) a_j u_j$

Linearization [Chew, 1999, Bleistein, 2012]

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The sparse spikes problem [De Castro and Gamboa, 2012, Bredies and Pikkarainen, 2013]

Linearized far-field:

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$$y = \Phi^b \mu + w, \qquad \Phi^b : \mu \mapsto [(\mathcal{F}\mu)(\omega_j)]_{j=1}^{m}$$

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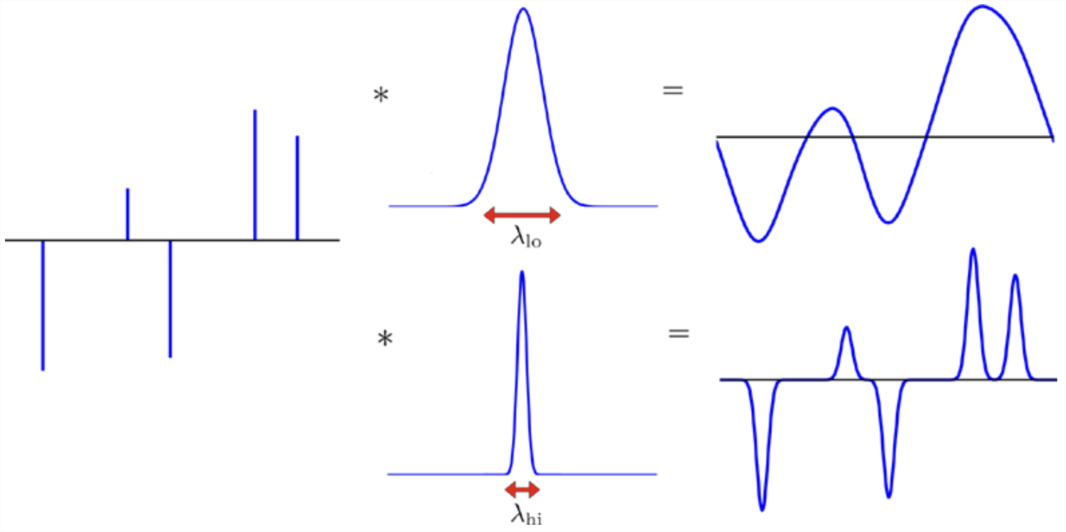
- Sample complexity results [Candès and Fernandez-Granda, 2014, Poon et al., 2023]:

$$m \gtrsim m(\Delta, s), \qquad \Delta = \min_{i,k} |x_i - x_k|$$

The super-resolution / sparse spikes problem [Candès and Fernandez-Granda, 2013]



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$$u^{\infty}(\hat{x}_j, \theta_j) \propto \sum_{i=1}^{s} a_i u_i e^{-i\kappa \hat{x}_j \cdot x_i} \quad \rightsquigarrow \quad \mu = \sum_{i=1}^{s} a_i \delta_{x_i}$$

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1. Sparse spike problem approach with linearized forward operator

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"Linearize and locally optimize" approach

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Contributions

- Reconstruction guarantees for output of linear step
- Numerical evidence of accurate recovery for "highly nonlinear settings"

Bounding the linearization error

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Linear regime (Born approx.)

Isolated inhomogeneities with low intensities

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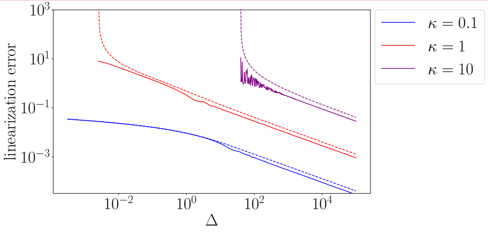
General bound

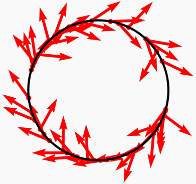
$$|u^\infty(\hat{x},\theta) - u^{\infty,b}(\hat{x},\theta)| \leq \frac{\kappa^2 \|a\|_1}{4\pi} \frac{\kappa^2 \phi(\Delta)\|a\|_1}{1-\kappa^2\phi(\Delta)\|a\|_1}, \qquad \phi(t) \propto \begin{cases} 1/\sqrt{t} & \text{if } d=2 \\ 1/t & \text{if } d=3 \end{cases}$$



Stable recovery

Sampling scheme

- $\omega_j \sim \mathcal{U}(B(0, 2\kappa)), \qquad j = 1, \ldots, m$
- $\kappa(\hat{x}_j - \theta_j) = \omega_j$



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Proposition (based on [Poon, Keriven and Peyré, 2023])

If

$$\Delta \gtrsim \frac{s^{2/(d+1)}}{\kappa} \quad \text{and} \quad m \gtrsim s \cdot \text{log factors}$$

then $\hat{\mu}$ is $\sqrt{s}\,\|w\|_2$-close to μ for $\lambda \sim \|w\|_2/\sqrt{s}$ with overwhelming probability.

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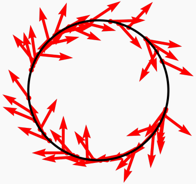

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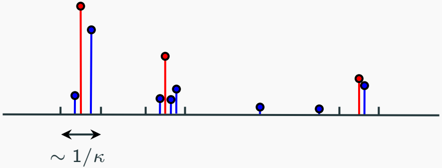

Exact support recovery

Proposition (based on [Poon, Keriven and Peyré, 2019])

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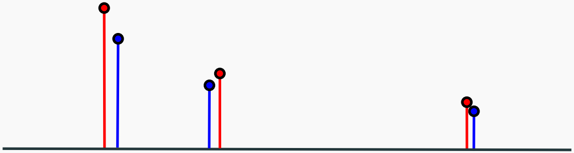

Reconstruction algorithm: linear step

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$$\min_{\mu} \frac{1}{2}\|\Phi^b \mu - y\| + \lambda|\mu|_{TV}, \qquad \Phi^b : \mu \mapsto [\mathcal{F}(\mu)(\kappa(\hat{x}_j - \theta_j))]_{j=1}^{m}$$

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Sliding Frank-Wolfe algorithm

- $\eta^{[k]} = -\frac{1}{\lambda}(\Phi^b)^*(\Phi^b \mu^{[k]} - y)$
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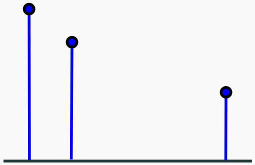

[Denoyelle et al., 2019]

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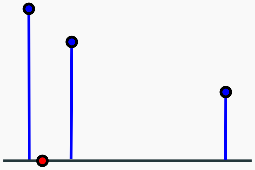

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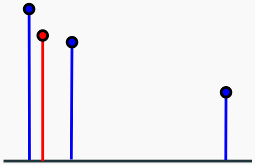

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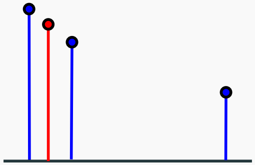

[Denoyelle et al., 2019]

Reconstruction algorithm: linear step

Variational problem

$$\min_{\mu} \frac{1}{2}\|\Phi^b \mu - y\| + \lambda|\mu|_{TV}, \qquad \Phi^b : \mu \mapsto [\mathcal{F}(\mu)(\kappa(\hat{x}_j - \theta_j))]_{j=1}^{m}$$

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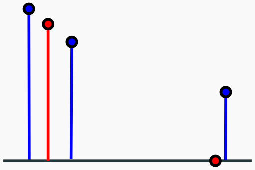

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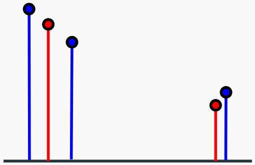

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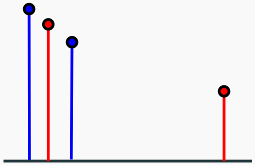

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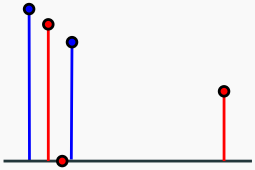

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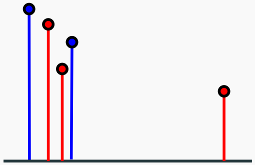

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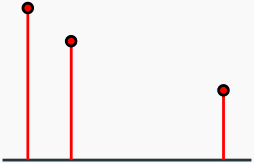

[Denoyelle et al., 2019]

Reconstruction algorithm: nonlinear step

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Objective

$$(a, x) \mapsto \frac{1}{2}\|\Phi_x a - y\| + \lambda\|a\|_1, \qquad \Phi_x : a \mapsto [u^\infty(\hat{x}_j, \theta_j)]_{j=1}^m$$

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“Nonlinear sliding”

Locally optimize

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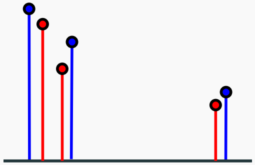

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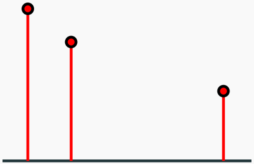
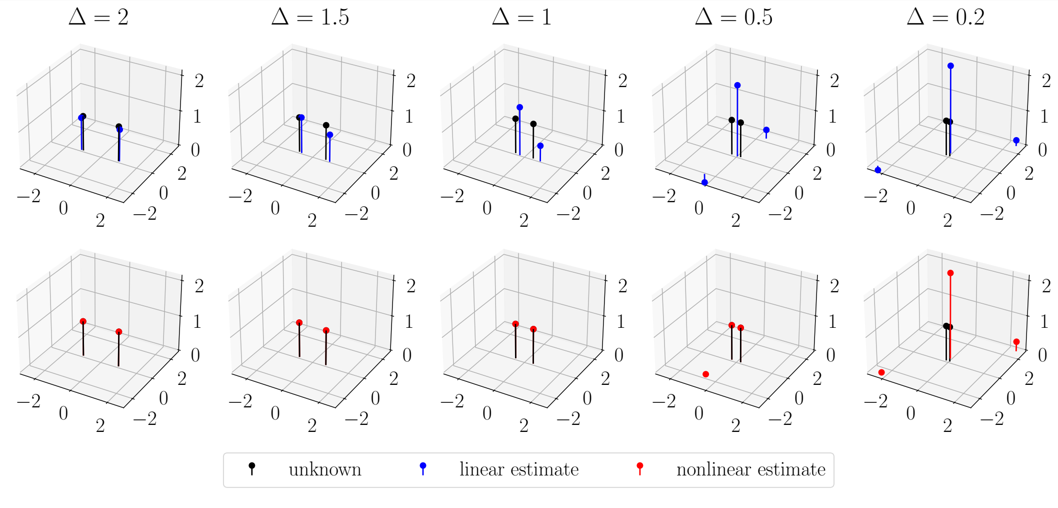
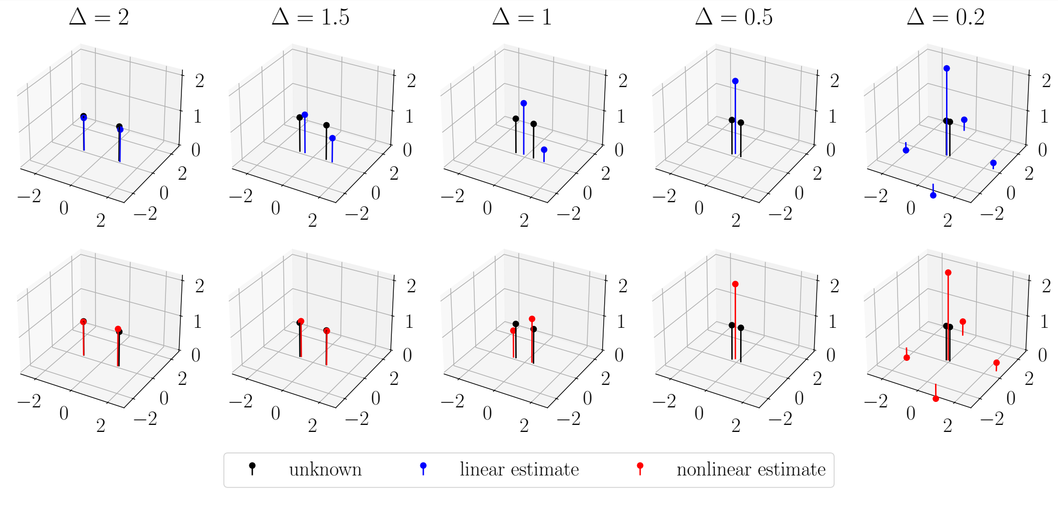


Numerical results: 2 spikes

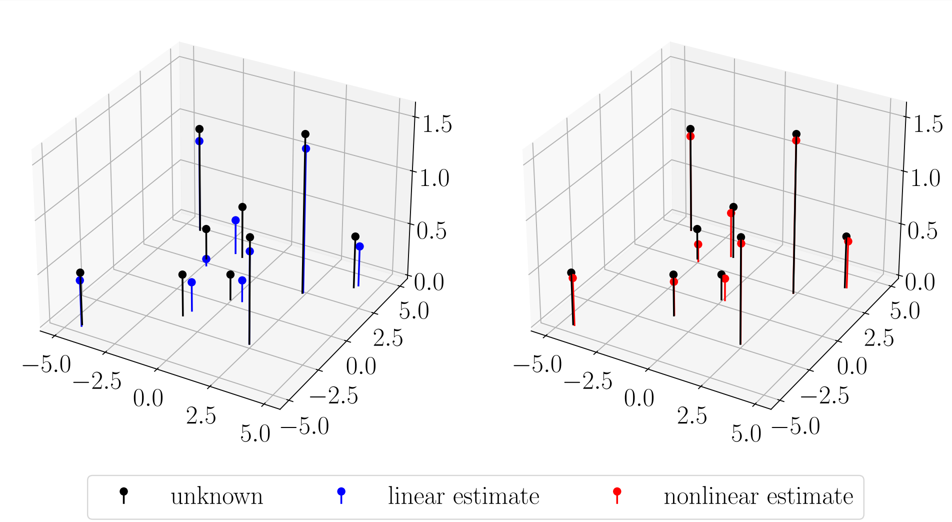


No measurement noise, $m = 20$

Numerical results: 2 spikes github.com/rpetit/pointscat

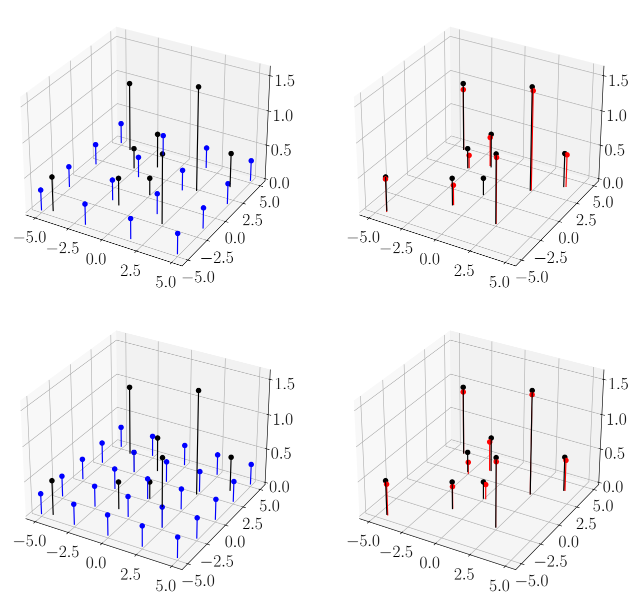


Gaussian noise $\sim 10\%$, $m = 20$



Gaussian noise $\sim 10\%$, $m = 100$

Numerical results: role of initialization



Conclusion

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Summary

- "Linearize and locally optimize" reconstruction method
- Reconstruction guarantees for output of linear step
- Numerical evidence of accurate recovery for "highly nonlinear settings"

A, R. Petit, and M. Santacesaria, *Localization of point scatterers via sparse optimization on measures*, SIAM Journal on Imaging Sciences 17.3 (2024): 1619-1649

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Perspectives

- More complex models for point-like sources
- Theoretical analysis of nonlinear step

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