Sampling in inverse problems

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Harmonic Analysis E-Seminars

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Joint work with

Alessandro Felisi
(UniGe)

Matteo Santacesaria
(UniGe)

S. Ivan Trapasso
(PoliTo)

Compressed sensing for inverse problems and the sample complexity of the sparse Radon transform, J. Eur. Math. Soc., to appear

Outline

Sampling and inverse problems

Compressed sensing

Compressed sensing for inverse problems

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Sampling

- Function $f \in L^2(\mathcal{D})$
- Sampling points $t_l \in \mathcal{D}$, $l = 1, \dots, m$
- Sampling problem:

$$(f(t_l))_{l=1}^{m} \rightsquigarrow f$$

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$$(f(t_l))_{l=1}^{m} \rightsquigarrow f$$

- Need assumptions on f
- Classically, f is ω-bandlimited (linear condition):

$$m \gtrsim \omega$$

Inverse problems

- $\mathcal{H}$ (e.g. $\mathcal{H} = L^2(\Omega)$): Hilbert space of inputs
- $F\colon \mathcal{H} \to L^2(\mathcal{D}; \mathcal{H}')$ linear **forward map** (compact)

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Examples

1. **Deconvolution** (with Bessel operator):

$$F = (I - \Delta)^{-b/2}\colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \qquad F(u) = \kappa_b * u$$

where $b > 2$ and $\kappa_b \coloneqq \mathcal{F}^{-1}\left((1 + |\cdot|^2)^{-b/2}\right)$

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2. **Radon transform**[1]:

$$\mathcal{R}\colon L^2(\mathcal{B}_1) \to L^2(\mathbb{S}^1; L^2(-1,1)), \qquad (\mathcal{R}u)(\theta) = \int_{\theta^\perp} u(y + \cdot\,\theta)dy \in L^2(-1,1)$$

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$$\left(F_{t_l} u := (Fu)(t_l)\right)_{l=1}^{m} \quad \rightsquigarrow \quad u$$

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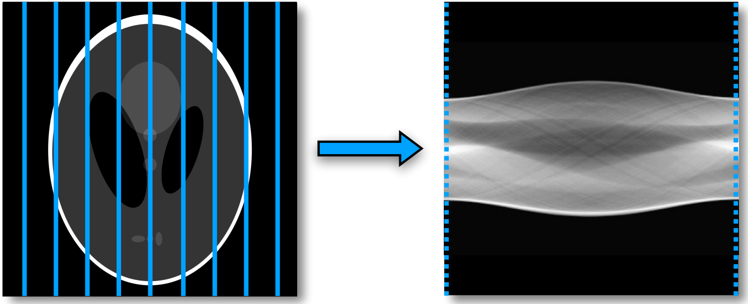
$$\left(\mathcal{R}_{\theta_l}u\right)_{l=1}^m \quad \rightsquigarrow \quad u$$

The sparse Radon transform

$$\mathcal{R}_\theta u(s) = \int_{\theta^\perp} u(y + s\theta)dy, \qquad \theta = \theta_1$$



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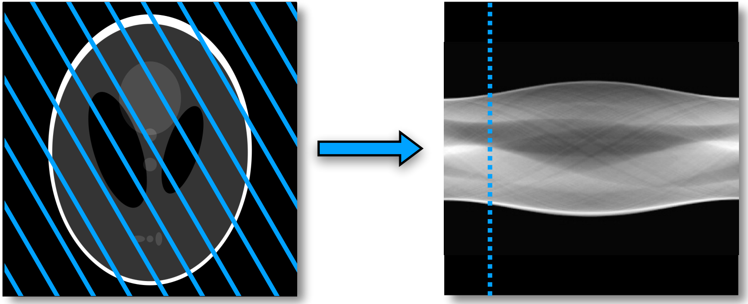
$$\mathcal{R}_\theta u(s) = \int_{\theta^\perp} u(y + s\theta)\,dy, \qquad \theta = \theta_2$$



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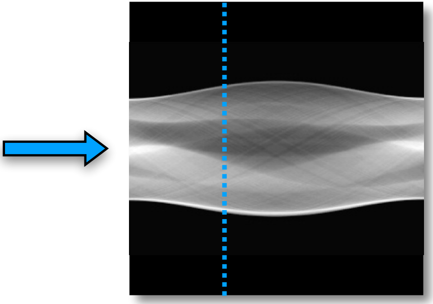
$$\mathcal{R}_\theta u(s) = \int_{\theta^\perp} u(y + s\theta)dy, \qquad \theta = \theta_3$$



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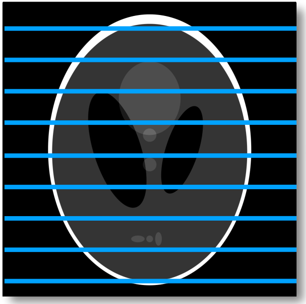

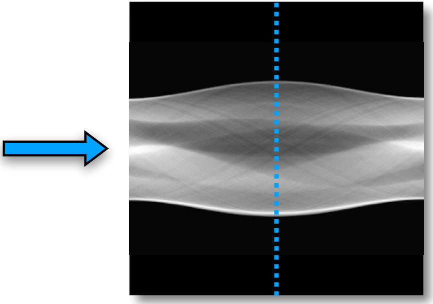

The sparse Radon transform

$$(\mathcal{R}u(\theta_1,\cdot),\dots,\mathcal{R}u(\theta_m,\cdot)), \quad \theta_1,\dots,\theta_m \in \mathbb{S}^1$$

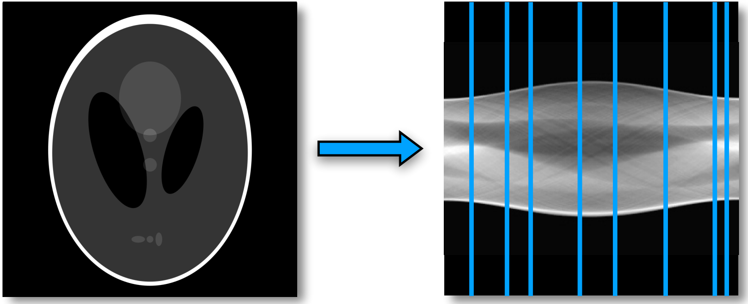

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Figure: **Important Genoese**

Figure: **Wavelet coefficients**

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- how to choose the samples $t_1, \ldots, t_m \in \mathcal{D}$
- and how many are needed ($m = ?$)

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An example: Magnetic Resonance Imaging

Measurements: $\mathrm{F} =$ Fourier transform

Compressed sensing[3]

Setup:

[3] E. J. Candès, J. K. Romberg, T. Tao. Stable signal recovery from incomplete and inaccurate measurements. Comm. Pure Appl. Math. 59(8) (2006), 1207-1223
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Setup:

- **Unknown**: $u^\dagger \in \mathbb{C}^M$ is s-sparse
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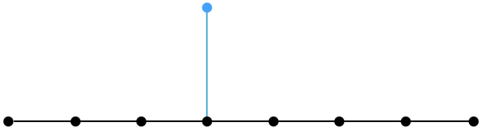

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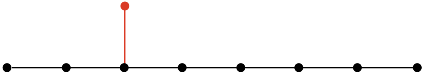

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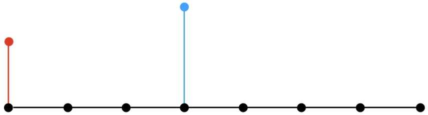

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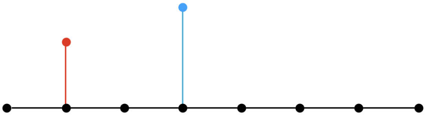

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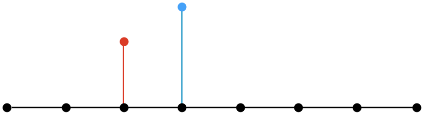

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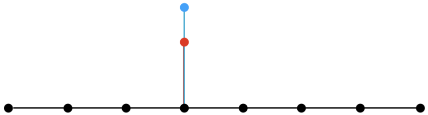

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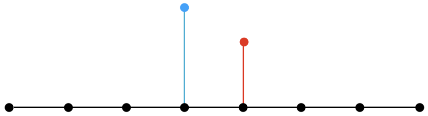

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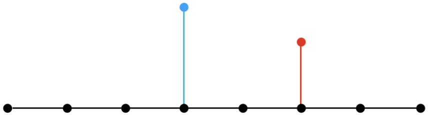

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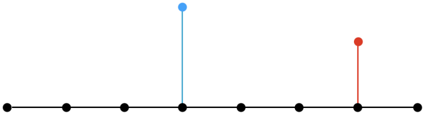

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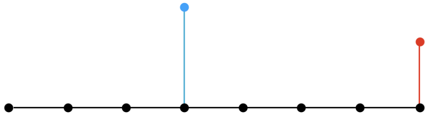

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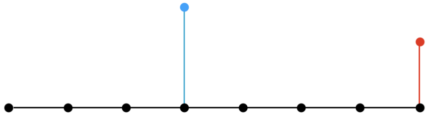

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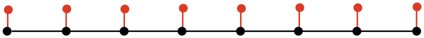

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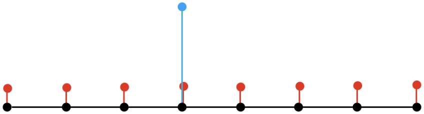

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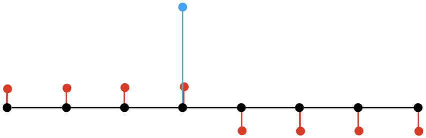

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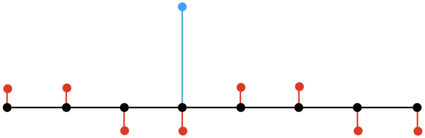

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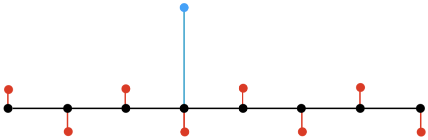

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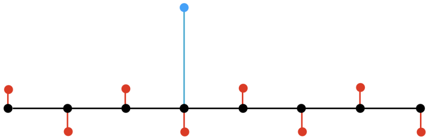

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This feature is measured by the coherence between the **sparsifying dictionary** and the **sensing system**

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$$B = \sqrt{M} \cdot \max_{n,t} |\langle \phi_n, \psi_t \rangle|$$

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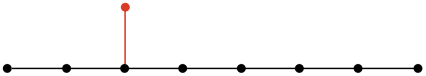

Figure: **First example**: $B = \sqrt{M}$

Coherence

$$B = \sqrt{M} \cdot \max |\langle \phi_n, \psi_t \rangle|$$

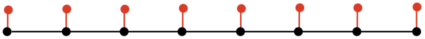

Figure: **Second example**: $B = 1$

Recovery estimate[4]

- $u^\dagger \in \mathbb{C}^M$ **unknown**
- $u^\dagger$ is s-**sparse** w.r.t. $\{\phi_n\}_{n=1}^M$
- **minimization problem**

$$\widehat{u} \in \arg\min_{u \in \mathbb{C}^M} \|(\langle u, \phi_n \rangle)_n\|_1 \quad : \quad \langle u, \psi_{t_l} \rangle = \langle u^\dagger, \psi_{t_l} \rangle, \; l = 1, \ldots, m$$

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Recovery estimate[4]

- $u^\dagger \in \mathbb{C}^M$ **unknown**
- $u^\dagger$ is s-**sparse** w.r.t. $\{\phi_n\}_{n=1}^M$
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If

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Outline

Sampling and inverse problems

Compressed sensing

Compressed sensing for inverse problems

Setting[5]

- $\mathcal{H} = L^2(\Omega)$ with $\Omega \subset \mathbb{R}^2$ bounded
- $(\phi_{j,n})_{j,n}$: sufficiently nice **wavelet basis**
- $u^\dagger$ is **s-sparse** w.r.t. the wavelet basis

[5] B. Adcock, A. C. Hansen, C. Poon, B. Roman, Breaking the coherence barrier: A new theory for compressed sensing, Forum Math. Sigma, 2017.
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Forward map

$$u \longmapsto (F_{t_l} u)_{l=1}^m$$

not a subsampled isometry, but a **subsampled compact** operator

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  Compressive sensing connects the critical number of projections to the image sparsity, but does not cover CT. Empirical results suggest a similar connection.

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Key tools:

1. Quasi-diagonalization of F
2. Relative coherence

Quasi-diagonalization

- $u^\dagger$ is **s-sparse**

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- **pseudo-sparsity property** on $Fu^\dagger$: use compressed sensing methods

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Relative coherence

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Classically defined as

$$B = \sqrt{M} \sup_{n,t} |F_t(\phi_n)|$$

when we sampled with respect to the **uniform probability** on $[M]$

Relative coherence

$$B = \sqrt{M} \sup_{n,t} |F_t(\phi_n)|$$

$g(t) \equiv 1/M$ is the **probability density** w.r.t. the counting measure on $\mathcal{D} = \{1, \dots, M\}$

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Sampling rule in general $\mathcal{D}$ (e.g. $\mathcal{D} \subseteq \mathbb{R}^d$):

$$g(t)\, dt$$

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We call this quantity relative coherence

Optimizing the probability density $g(t)$

The **optimal choice** for $g(t)$ (minimizing B_{rel}) depends on the decay in t of $|F_t \phi_{j,n}|$

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Examples

1. **Deconvolution** for $b > 2$:

$$|\big((I-\Delta)^{-b/2}\phi_{j,n}\big)(t)| \lesssim \frac{e^{-C_b|d(t,\Omega)|}}{2^{(b-1)j}} \quad \Rightarrow \quad g(t) \propto e^{-C_b|d(t,\Omega)|}$$

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2. **The Radon transform**:

$$\|\mathcal{R}_\theta \phi_{j,n}\|_{L^2([-1,1])} \lesssim 1 \quad \Rightarrow \quad g(\theta) = \frac{1}{2\pi}$$

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Theorem

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A corollary for the Radon transform[8]

$$\big(\mathcal{R}u^\dagger(\theta_1,\cdot),\dots,\mathcal{R}u^\dagger(\theta_m,\cdot)\big)\in L^2(-1,1)^m \qquad \longrightarrow \qquad u^\dagger\in L^2(\mathcal{B}_1)$$

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Conclusions

Past

- Theory of CS for random matrices and subsampled isometries (e.g. MRI)
- Empirical evidence for compressed sensing Radon transform

Present

- Abstract theory of sample complexity for inverse problems
- Rigorous theory of compressed sensing for the sparse Radon transform

Future

- Wavelets $\rightarrow$ shearlets, curvelets, etc.
- Generalisation to other ill-posed problems, possibly nonlinear

Paper

Slides

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