Non-zero Constraints in PDEs and Applications to Hybrid Inverse Problems

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Internal data in quantitative hybrid imaging problems

- Hybrid conductivity imaging [Widlak, Scherzer, 2012]

$$\begin{cases} -\mathrm{div}(a\,\nabla u_i) = 0 & \text{in } \Omega, \\ u_i = \varphi_i & \text{on } \partial\Omega. \end{cases}$$

$$u_i(x) \quad \text{or} \quad a(x)\,\nabla u_i(x) \quad \text{or} \quad a(x)\,|\nabla u_i|^2\,(x) \qquad \xrightarrow{?} \qquad a$$

- Quantitative thermoacoustic tomography [Bal et al., 2011, Ammari et al., 2013]

$$\begin{cases} \Delta u_i + (\omega^2 + \mathrm{i}\omega\sigma)\,u_i = 0 & \text{in } \Omega, \\ u_i = \varphi_i & \text{on } \partial\Omega. \end{cases}$$

$$\sigma(x)\,|u_i|^2\,(x) \qquad \xrightarrow{?} \qquad \sigma$$

- MREIT [Seo et al., 2012, Bal and Guo, 2013]

$$\begin{cases} \mathrm{curl} E^i = \mathrm{i}\omega H^i & \text{in } \Omega, \\ \mathrm{curl} H^i = -\mathrm{i}(\omega\varepsilon + \mathrm{i}\sigma) E^i & \text{in } \Omega, \\ E^i \times \nu = \varphi_i \times \nu & \text{on } \partial\Omega. \end{cases}$$

$$H^i(x) \qquad \xrightarrow{?} \qquad \varepsilon, \sigma$$

Internal data in quantitative hybrid imaging problems

- Hybrid conductivity imaging [Widlak, Scherzer, 2012], quantitative photoacoustic tomography

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Non-vanishing gradients and Jacobians

- Consider for simplicity the hybrid conductivity problem with internal data ∇u and unknown a:

$$\begin{cases} -\mathrm{div}(a\,\nabla u) = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

- With 1 measurement:

$$\nabla a \cdot \nabla u = -a\Delta u \qquad \Longrightarrow \qquad \nabla(\log a)\cdot \nabla u = -\Delta u$$

This equation may be solved in a if a is known on $\partial\Omega$ and if

$$\nabla u(x) \neq 0, \qquad x \in \Omega.$$

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Main question

Is it possible to find suitable boundary values φ_i so that the corresponding solutions u_i satisfy certain non-zero constraints, such as a non-vanishing Jacobian

$$\det \begin{bmatrix} \nabla u_1(x) & \cdots & \nabla u_d(x) \end{bmatrix} \neq 0\,?$$

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Outline of the talk

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The Radó-Kneser-Choquet theorem

Theorem (Alessandrini Magnanini 1994, Bauman et al. 2000, Alessandrini Nesi 2015)

Let $\Omega \subseteq \mathbb{R}^2$ be bounded and convex and $a \in C^{0,\alpha}(\overline{\Omega}; \mathbb{R}^{2\times 2})$ be uniformly elliptic. Let $u_i \in H^1(\Omega)$ solve

$$-\mathrm{div}(a\nabla u_i) = 0 \qquad \text{in } \Omega, \qquad\qquad u_i = x_i \qquad \text{on } \partial\Omega.$$

Then

$$\det \begin{bmatrix} \nabla u_1(x) & \nabla u_2(x) \end{bmatrix} \neq 0, \qquad x \in \Omega.$$

- $\alpha\nabla u_1(x_0) + \beta\nabla u_2(x_0) = 0$
- Set $v(x) = \alpha u_1(x) + \beta u_2(x)$:
- $\nabla v(x_0) = 0$
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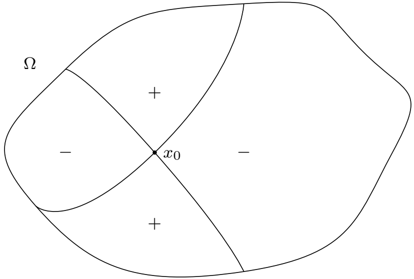

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The failure in 3D and for other elliptic PDEs

- In 3D, the above result fails. Counterexamples by Laugesen 1996, Briane et al 2004 and Capdeboscq 2015: it is not possible to find $(\varphi^1, \varphi^2, \varphi^3)$ independently of a so that

$$\det \begin{bmatrix} \nabla u_1(x) & \nabla u_2(x) & \nabla u_3(x) \end{bmatrix} \neq 0, \qquad x \in \Omega.$$

- This result fails for Helmholtz type problems or for eigenvalue problems

$$\operatorname{div}(a\nabla u) + k^2 q u = 0$$

since solutions oscillate.

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Critical points in 3D

What about critical points: can we find φ independently of a so that

$$\nabla u(x) \neq 0, \qquad x \in \Omega?$$

Theorem (GSA, Bal, Di Cristo, ARMA 2017)

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded Lipschitz domain. Take $\varphi \in C(\partial X) \cap H^{\frac{1}{2}}(\partial X)$. There exists a (nonempty open set of) $a \in C^\infty(\overline{X})$ such that the solution $u \in H^1(X)$ to

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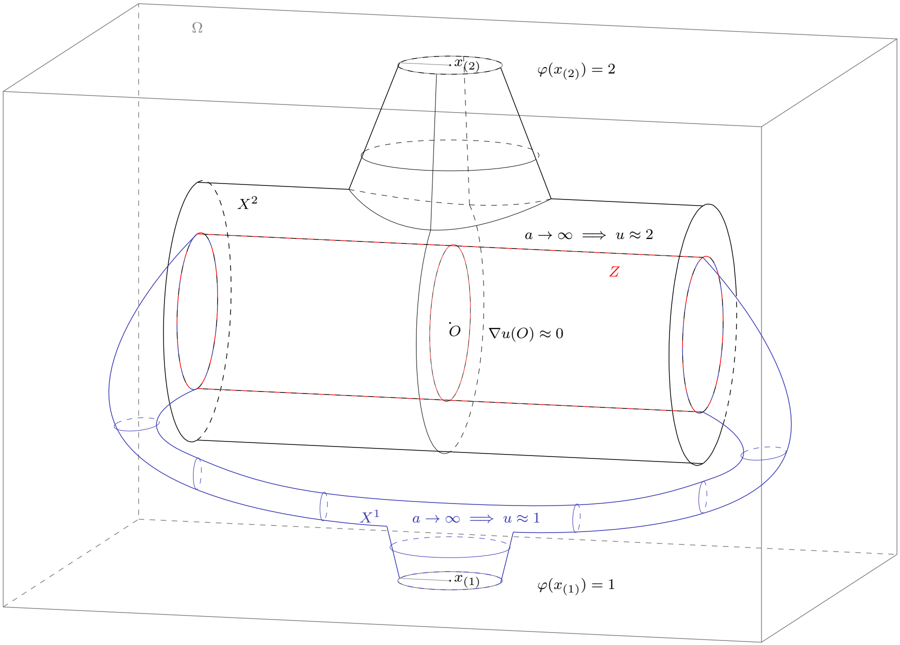
Ω
$x_{(2)}$
$\varphi(x_{(2)}) = 2$
X^2
$a \to \infty \implies u \approx 2$
Z
O
$\nabla u(O) \approx 0$
X^1
$a \to \infty \implies u \approx 1$
$x_{(1)}$
$\varphi(x_{(1)}) = 1$

Alternative approaches

- Complex geometrical optics solutions [Bal and Uhlmann, IP 2010]

$$\cos^2 x + \sin^2 x = 1 \neq 0$$

- Use multiple frequencies [GSA, Commun. PDE 2015]

$$\cos^2(k_1 x) + \cos^2(k_2 x) \neq 0$$

Also with Neumann eigenfunctions (bypassing the "hot spots" conjecture):

$$\sum_{k=1}^{K} |\nabla e_k(x)|^2 \geq c, \qquad x \in \overline{\Omega'}$$

[GSA, Barnes, Jambhale and Nickl, Math. Stat. Learn. 2025]

- Runge approximation & Whitney embedding
- Random boundary conditions

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The model problem

- Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$ be a smooth bounded domain. Consider the elliptic PDE

$$Lu := -\mathrm{div}(a\nabla u) + b \cdot \nabla u + cu = 0 \quad \text{in } \Omega,$$

  with a, b and c smooth enough so that $u \in C^{1,\alpha}$ and the unique continuation property (UCP) holds

- Consider, for simplicity, the non-vanishing Jacobian constraint: look for φ_i such that

$$\det \begin{bmatrix} \nabla u_1 & \cdots & \nabla u_d \end{bmatrix} (x) \neq 0$$

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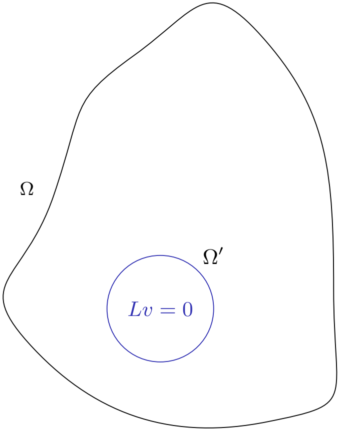
Main tool: the Runge Approximation [Lax 1956]



- Let $\Omega' \subseteq \Omega$ be simply connected and $v \in H^1(\Omega')$ be a local solution:
$$Lv = 0 \quad \text{in } \Omega'.$$

  In general, v cannot be extended to a global solution u, BUT:

- Runge approximation: there exist global solutions u_n to
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  such that
$$\| u_n|_{\Omega'} - v \|_{L^2(\Omega')} \to 0.$$

- By elliptic regularity, we get for $\Omega'' \Subset \Omega'$:
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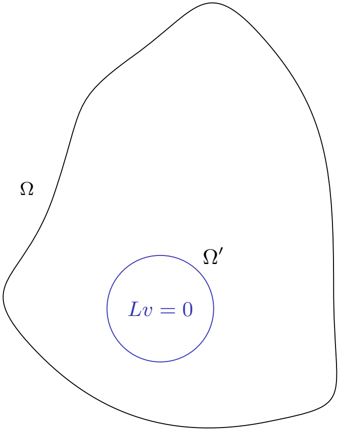

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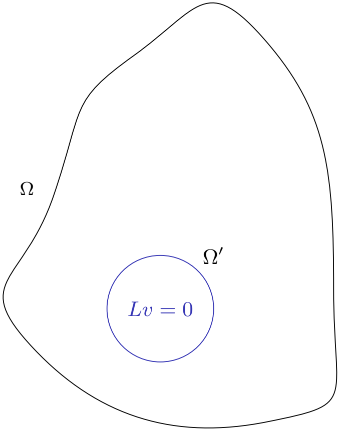

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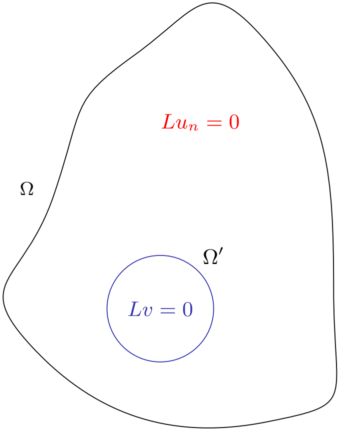

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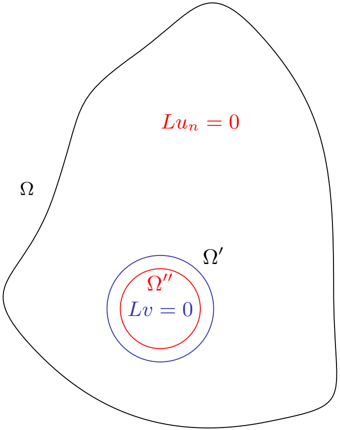

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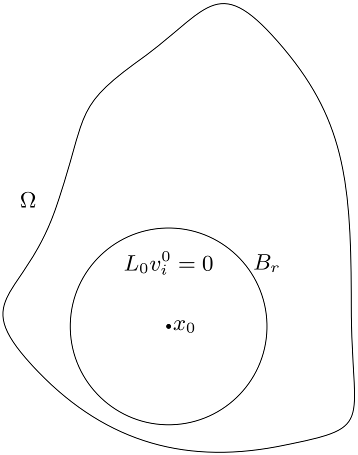
The Runge approximation and non-zero constraints [Bal and Uhlmann, CPAM 2013]



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4. Covering of $\overline{\Omega}$ with N balls: $N \cdot d$ boundary conditions.

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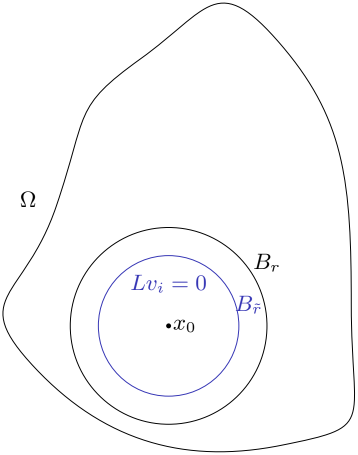

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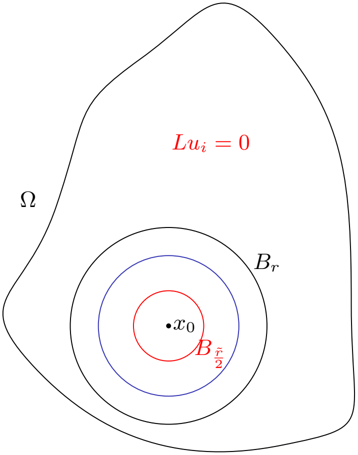

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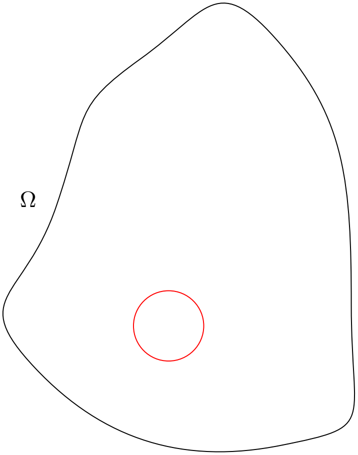

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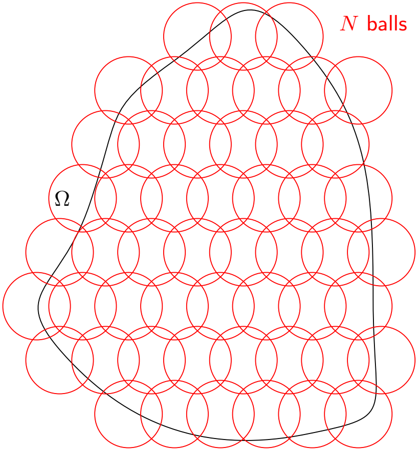

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Two main issues

- You need a large number of measurements to satisfy the constraint

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Whitney projection argument

Lemma (Greene and Wu 1975)

Take $k > 2d$ (possibly large). Let $u_1, \dots, u_k$ be solutions to $Lu_i = 0$ in Ω such that

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Then, for almost every $a \in \mathbb{R}^{k-1}$, we have

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In other words: we can almost always reduce the number of solutions (until $2d$) and keep the constraint.
In particular, arbitrarily small weights a can be used.

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Lemma (Greene and Wu 1975)

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Runge & Whitney: main result

Theorem (GSA and Capdeboscq, IMRN 2019)

The set of $2d$ solutions $u_1, \dots, u_{2d}$ to $Lu_i = 0$ in Ω such that

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is open and dense in the set of $2d$ solutions to $Lu_i = 0$ in Ω.

Proof.
Open. The rank is stable under small perturbations of u_i.
Dense. Take $\tilde{u}_1, \dots, \tilde{u}_{2d}$ solutions to $L\tilde{u}_i = 0$. By Runge, we have a large number of solutions so that

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Remarks on the result

- As a corollary, the set of $2d$ boundary conditions whose solutions satisfy the constraint everywhere is open and dense.
- The approach is very general, and works with many other constraints, like

$$\begin{array}{ll} |u_1|(x) > 0 \text{ (nodal set)} & d+1 \text{ solutions} \\ \left|\det \begin{bmatrix} \nabla u_1 & \cdots & \nabla u_d \end{bmatrix}\right|(x) > 0 \text{ (Jacobian)} & 2d \text{ solutions} \\ \left|\det \begin{bmatrix} u_1 & \cdots & u_{d+1} \\ \nabla u_1 & \cdots & \nabla u_{d+1} \end{bmatrix}\right|(x) > 0 \text{ (``augmented'' Jacobian)} & 2d+1 \text{ solutions} \end{array}$$

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Outline of the talk

Using random boundary conditions

- ν: subgaussian probability distribution on $H^{1/2}(\partial\Omega)$
- $\varphi \sim \nu$ is of the form

$$\varphi = \sum_{k\in\mathbb{N}} a_k e_k$$

where $\{e_k\}$ is an ONB of $H^{1/2}(\partial\Omega)$ and $\{a_k\}_k$ are uncorrelated real random variables

Theorem (GSA, IP 2022)

Take $N \in \mathbb{N}$. Let $\varphi_1^l, \dots, \varphi_d^l \sim \nu$ be sampled i.i.d. in $H^{1/2}(\partial\Omega)$ for $l = 1, \dots, N$. Then

$$\max_{l=1,\dots,N} \left|\det \begin{bmatrix} \nabla u_1^l & \cdots & \nabla u_d^l \end{bmatrix}(x)\right| \geq C_1, \qquad x \in \overline{\Omega'},$$

where

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Two steps:

1. By quantitative Runge approximation (as in Salo-Rüland-2018, but with arbitrary norms on $\partial\Omega$):

$$\mathbb{E}(\left| \det \begin{bmatrix} \nabla u_1 & \cdots & \nabla u_d \end{bmatrix} (x) \right|) \geq C$$

2. Concentration inequalities

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Theorem (GSA, IP 2022)

Take $N \in \mathbb{N}$. Let $\varphi_1^l, \dots, \varphi_d^l \sim \nu$ be sampled i.i.d. in $H^{1/2}(\partial\Omega)$ for $l = 1, \dots, N$. Then

$$\max_{l=1,\dots,N} \left|\det \begin{bmatrix} \nabla u_1^l & \cdots & \nabla u_d^l \end{bmatrix} (x)\right| \geq C_1, \qquad x \in \overline{\Omega'}$$

with probability greater than

$$1 - C_2 N^{d/2} \exp(-C_3 N^{1/d}).$$

Two steps:

1. By quantitative Runge approximation (as in Salo-Rüland-2018, but with arbitrary norms on $\partial\Omega$):

$$\mathbb{E}(\left|\det \begin{bmatrix} \nabla u_1 & \cdots & \nabla u_d \end{bmatrix} (x)\right|) \geq C$$

2. Concentration inequalities

Random boundary values: simulations [GSA, Cen and Zhou, preprint 2025]

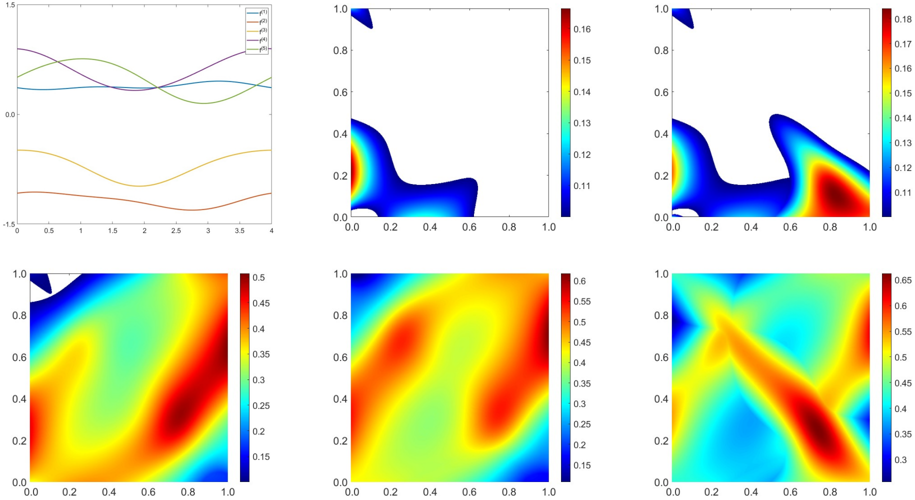

Figure: Boundary values and the non-zero region $\max_{\ell=1,\dots,N} |\partial_{x_1} u^{(\ell)}(x)| \geq 0.1$, $N = 1, 2, 3, 4, 5$.

Conclusions

- The inversion in quantitative hybrid imaging often requires the solutions to the direct problem to satisfy certain non-zero constraints.
- Available methods:
  - The Radó-Kneser-Choquet theorem and its generalizations (only in 2D, not for Helmholtz)
  - CGO solutions
  - multiple frequencies
  - Runge & Whitney
  - Random boundary values
- Future prospectives:
  - combine Runge & Whitney with random boundary values
  - other PDEs (Maxwell, elasticity, etc.)
  - numerical experiments

Slides: